University of Southern Mindanao

Linear Algebra

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Workbook

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Linear Algebra Workbook Writer:
Philip Lester P. Benjamin, PhD

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Chapter 1

Systems of Linear Equation

One of the most frequently recurring practical problems in many fields of study such as mathematics, physics, biology, chemistry, economics, all phases of engineering, operations research, and the social science is that of solving a system of linear equations.

Linear Equation

A **linear equation** is of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, where x_i's are unknowns and a_i's are real or complex numbers. A **solution** to this linear equation is a sequence $< s_n >$ of n numbers $s_1, s_2, \ldots, s_n$ such that $x_i = s_i$ for each i.

EXAMPLE 1. Consider the linear equation $6x_1 - 3x_2 + 4x_3 = -13$. Set $x_1 = 2, x_2 = 3, x_3 = -4$. Then

$$6x_1 - 3x_2 + 4x_3 = 6(2) - 3(3) + 4(-4) = -13.$$

This means that $x_1 = 2, x_2 = 3, x_3 = -4$ is a solution to the given linear equation.

System of Linear Equations

A **system of m linear equations in n unknowns** or a **linear system** is a set of m linear equations each in n unknowns. A linear system is of the form

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\quad \vdots \quad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

A solution to a linear system is a sequence $< s_n >$ of n numbers such that each linear equations is satisfied when $x_i = s_i$ for each i.

If a linear system has no solution, then it is said to be **inconsistent**.

If it has a solution, then it is called **consistent**.

If $b_i = 0$ for each i in each linear equation in a linear system, then the linear system is called a **homogeneous system**.

Remark: If $x_i = 0$ for each i in a homogeneous linear system, then the system is satisfied. This solution is called a **trivial solution**.

A sequence $< s_n >$ is a nontrivial solution to a homogeneous linear system if $s_i \neq 0$ for all i.

Two linear systems are **equivalent** if they both have exactly the same solutions.

EXAMPLE 2. The linear systems

$$\begin{aligned} x_1 - 3x_2 &= -7 \\ 2x_1 + x_2 &= 7 \end{aligned}$$

and

$$\begin{aligned} 8x_1 - 3x_2 &= 7 \\ 3x_1 - 2x_2 &= 0 \\ 10x_1 - 2x_2 &= 14 \end{aligned}$$

are equivalent. They both have $x_1 = 2$ and $x_2 = 3$ as a solution.

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1. Solve the given linear system by method of elimination.

$$\begin{aligned} x + 2y &= 8 \\ 3x - 4y &= 4 \end{aligned}$$

2. Given a linear system

$$\begin{aligned} 2x - y &= 5 \\ 4x - 2y &= t \end{aligned}$$

   (a) Determine a particular value of t so that the system is consistent.

   (b) Determine a particular value of t so that the system is inconsistent.

Another method to solve the solution of a linear system is through the use of matrices.

Definition of a Matrix

An $m \times n$ **matrix** A is a rectangular array of mn real or complex numbers arranged in m horizontal rows and n vertical columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Square Matrix

If $m = n$, we sya that A is a **square matrix** of order n.
The numbers $a_{11}, a_{22}, \cdots, a_{nn}$ form the **main diagonal** of A.

Notation for a matrix: $A = [a_{ij}]$, where a_{ij} is the entry of A in the ith row and jth column.

EXAMPLE 3.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix}$$

A is a 3×3 matrix or a square matrix of order 3, $a_{23} = 1$,

Diagonal entries are $a_{11} = 1, a_{22} = -1, a_{33} = -1$

n-Vector

An $n \times 1$ matrix is called an n-vector.

We will discuss more of vectors on later part of this course.

EXAMPLE 4.

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

is a 3-vector.

The set of all n-vectors with real entries is denoted by $\mathbb{R}^n$, i.e,

$$\mathbb{R}^n = \{\vec{u} \mid \vec{u} \text{ is an } n\text{-vector with real entries}\}.$$

An n-vector whose entries are all zero is called a **zero vector** and is denoted by $\vec{0}$.

Equal Matrices

Two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** if $a_{ij} = b_{ij}$ for all i and j.

EXAMPLE 5.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 0 & -1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} x & 2 & 3 \\ 2 & -1 & y \\ 3 & 0 & -1 \end{bmatrix}$$

are equal if $x = 1$ and $y = 4$.

1.1 Matrix Operations

Sum of Matrices

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices, then the **sum** $A + B$ is an $m \times n$ matrix $C = [c_{ij}]$ defined by $c_{ij} = a_{ij} + b_{ij}$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

EXAMPLE 6. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 3 \\ 3 & 1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 6 \\ 5 & 0 & 8 \\ 6 & 1 & -2 \end{bmatrix}$

Scalar Multiple of Matrices

If $A = [a_{ij}]$ is an $m \times n$ matrix and r is a real number, then the **scalar multiple** of A by r, denoted by rA, is the $m \times n$ matrix $C = [c_{ij}]$, such that $c_{ij} = ra_{ij}$, for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

EXAMPLE 7. $2 \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 4 & -2 & 8 \\ 6 & 0 & -2 \end{bmatrix}$

Difference of Matrices

If A and B are $m \times n$ matrices, we write $A + (-1)B = A - B$. We call this the **difference** between A and B.

EXAMPLE 8. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 0 & -1 \end{bmatrix}$ - $\begin{bmatrix} 1 & -2 & 3 \\ 3 & 1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ 1 & -2 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

Dot Product

Let $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be n-vectors in $\mathbb{R}^n$.

The **dot product** or **inner product** of vectors $\vec{a}$ and $\vec{b}$ is defined as

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i$$

EXAMPLE 9. The dot product of $\vec{a} = \begin{bmatrix} 1 \\ -3 \\ 4 \\ 3 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ -2 \end{bmatrix}$ is

$$\vec{a} \cdot \vec{b} = 1(2) + (-3)(3) + 4(1) + 3(-2) = -9.$$

Product of Matrices

If $A = [a_{ij}]$ is an $m \times p$ matrix and $B = [b_{ij}]$ is an $p \times n$ matrix, then the **product** of A and B, denoted by AB, is the matrix $C = [c_{ij}]$ defined by

$$c_{ij} = \sum_{i=1}^{p} a_{ik} b_{kj}$$

EXAMPLE 10. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 5 \\ 4 & -3 \\ 2 & 1 \end{bmatrix}$, then $AB = \begin{bmatrix} 4 & -2 \\ 6 & 16 \end{bmatrix}$

REMARK:

Matrix multiplication is not commutative. To see this, let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 7 \\ -1 & 3 \end{bmatrix}$. This means that $AB \neq BA$.

Transpose of a Matrix

If $A = [a_{ij}]$ is an $m \times p$ matrix, then the **transpose** of A, denoted by $A^T = [a_{ij}^T]$, is the $n \times m$ matrix defined by $a_{ij}^T = a_{ji}$.

EXAMPLE 11. If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ -1 & 4 \end{bmatrix}$

EXAMPLE 12. Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 3 & -1 \\ -2 & 2 \end{bmatrix}$.
Compute the $(3, 2)$ entry of AB.

Note that if $AB = C$, then the $(3, 2)$ entry of AB is c_{32}. This is $(row_3(A))^T \cdot col_2(B)$. Thus, $(row_3(A))^T \cdot col_2(B) = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = -5$

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1. Prove: If $\vec{u}$ and $\vec{v}$ are n-vectors, then $[\vec{u}\cdot\vec{v}] = (\vec{u})^T\vec{v}$.

2. Let $A = \begin{bmatrix} 3 & 2 & 3 \\ 4 & 7 & 1 \\ 8 & 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 4 \\ 3 & 5 \\ 2 & 2 \end{bmatrix}$.
   Compute the $(3, 1)$ entry of AB.

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Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 1 \\ 5 & -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 & 2 \\ 3 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 0 & 1 \\ 5 & 4 & 2 \\ 3 & 2 & 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

1. Perform the following operations(if possible):

   (a) $A + B$

   (b) $2(A - C)$

   (c) $(AB)^T$

   (d) $\vec{u} \cdot \vec{v}$

Now we transform a given linear system to a matrix form. Consider the linear system of m equations and n unknowns.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Set $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Then

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \\ &= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} \end{aligned}$$

Observe that the entries of $A\vec{x}$ are the left side of the equations of the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Thus, we can express

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

as $A\vec{x} = \vec{b}$.

The matrix A is called the **coefficient matrix** of the linear system and the matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the **augmented matrix** of the linear system.

REMARK:

A homogeneous linear system can be written as $A\vec{x} = \vec{0}$.

EXAMPLE 13. Consider the linear system

$$\begin{aligned} -5x + z &= 0 \\ x + 2y - 4z &= 7 \\ 3x - 2y + z &= 3 \end{aligned}$$

Setting $A = \begin{bmatrix} -5 & 0 & 1 \\ 1 & 2 & -4 \\ 3 & -2 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix}$,

we can express the given linear system as $A\vec{x} = \vec{b}$.

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Consider the linear system

$$\begin{aligned} x + y + 2z &= -1 \\ x - 2y + 2z &= -5 \\ 3x + y + z &= 3 \end{aligned}$$

Determine the coefficient matrix of the given linear system and express this in matrix form.

Chapter 2

Algebraic Properties of Matrix Operations

2.1 Properties of Matrix Operations

We now consider the properties of matrix operations. We observe here that some properties of matrix operations behave differently when compared to the operations on the set of real numbers.

Properties of Matrices under Addition

THEOREM 1. Let A, B, C be $m \times n$ matrices. Then

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. There exists a unique $m \times n$ matrix O such that $A + O = A$ for any $m \times n$ matrix A. Here, matrix O is called the $m \times n$ **zero matrix**.
4. For all $m \times n$ matrix A, there exists a unique $m \times n$ matrix D such that $A + D = O$. In this case, we write $D = -A$. The matrix $-A$ is called the **negative** of A and note that $-A = (-1)A$.

Proof:

1 Let $A = [a_{ij}], B = [b_{ij}], A + B = C = [c_{ij}]$, and $B + A = D = [d_{ij}]$.
We claim that $c_{ij} = d_{ij}$ for all i, j.
Now, $c_{ij} = a_{ij} + b_{ij}$ and $d_{ij} = b_{ij} + a_{ij}$ for all i, j. Because $a_{ij}, b_{ij} \in \mathbb{R}$, commutativity under addition holds so that $a_{ij} + b_{ij} = b_{ij} + a_{ij}$. This implies that $c_{ij} = d_{ij}$. This proves the claim. Consequently, $A + B = B + A$.

2 Let

$$A = [a_{ij}], B = [b_{ij}], C = [c_{ij}], B + C = D = [d_{ij}], A + B = E = [e_{ij}],$$

$$A + (B + C) = F = [f_{ij}], (A + B) + C = G = [g_{ij}].$$

We show that $f_{ij} = g_{ij}$ for all i, j.

Now, $d_{ij} = b_{ij} + c_{ij}$ and $e_{ij} = a_{ij} + b_{ij}$. It follows that

$$\begin{aligned} f_{ij} &= a_{ij} + d_{ij} \\ &= a_{ij} + (b_{ij} + c_{ij}) \\ &= (a_{ij} + b_{ij}) + c_{ij} \text{ (since } \mathbb{R} \text{ is associative under usual addition)} \\ &= g_{ij}. \end{aligned}$$

Therefore, $A + (B + C) = (A + B) + C$.

The proofs for 3 and 4 will be left as an exercise. ■

Properties of Matrices under Multiplication

THEOREM 2. If A, B, C are matrices of appropriate sizes, then

1. $A(BC) = (AB)C$
2. $(A + B)C = AC + BC$
3. $C(A + B) = CA + CB$

Proof:

1 Suppose that A is $m \times n$, B is $n \times p$, and C is $p \times q$. Let $A = [a_{ij}], B = [b_{ij}], C = [c_{ij}], AB = D = [d_{ij}], BC = E = [e_{ij}], (AB)C = F = [f_{ij}]$, and $A(BC) = G = [g_{ij}]$. We will show that $f_{ij} = g_{ij}$ for all i, j. Now,

$$f_{ij} = \sum_{k=1}^{p} d_{ik} c_{kj} = \sum_{k=1}^{p} \left(\sum_{r=1}^{n} a_{ir} b_{rk} \right) c_{kj}.$$

Thus,

$$\begin{aligned} f_{ij} &= \sum_{k=1}^{p} \left(\sum_{r=1}^{n} a_{ir} b_{rk} \right) c_{kj} \\ &= \sum_{k=1}^{p} \left(a_{i1} b_{1k} + \cdots + a_{in} b_{nk} \right) c_{kj} \\ &= a_{i1} \sum_{k=1}^{p} b_{1k} c_{kj} + \cdots + a_{in} \sum_{k=1}^{p} b_{nk} c_{kj} \\ &= \sum_{r=1}^{n} a_{ir} \left(\sum_{k=1}^{p} b_{rk} c_{kj} \right) \end{aligned}$$

Similarly,

$$g_{ij} = \sum_{r=1}^{n} a_{ir} e_{rj} = \sum_{k=1}^{n} a_{ir} \left(\sum_{k=1}^{p} b_{rk} c_{kj} \right).$$

The proofs for 2 and 3 are left as an exercise.■

Properties of Scalar Multiplication

THEOREM 3. If r and s are real numbers and A and B are matrices of the appropriate sizes, then

1. $r(sA) = (rs)A$
2. $(r + s)A = rA + sA$
3. $r(A + B) = rA + rB$
4. $A(rB) = r(AB) = (rA)B$

Properties of Transpose

THEOREM 4. If r is a scalar and A and B are matrices of the appropriate sizes, then

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(AB)^T = B^T A^T$
4. $(rA)^T = rA^T$

2.2 Special Types of Matrices

Diagonal, Scalar, and Identity Matrix

An $n \times n$ matrix $A = [a_{ij}]$ is called a **diagonal matrix** if $a_{ij} = 0$ for $i \neq j$.
A **scalar matrix** is a diagonal matrix whose diagonal entries are equal.
In particular, the scalar matrix whose diagonal entries are all equal to 1 is called the **identity matrix**.

EXAMPLE 14. Diagonal matrix:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Scalar matrix:

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Identity matrix:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Remarks:

1. If A is any $n \times n$ matrix, then $AI_n = A$ and $I_n A = A$.
2. If A is a scalar matrix, then there exists $r \in \mathbb{R}$ such that $A = rI_n$.

We now define a similar concept of exponents in matrices.

Let A be a square matrix. The powers of a matrix, for p a positive integer, is defined as $A^p = \underbrace{A \cdot A \cdots A}_{p\ factors}$.

If A is $n \times n$, we define $A^0 = I_n$.

Some Results about the Powers of a Matrix

For nonnegative integers p and q, the following can be shown:

1. $A^p A^q = A^{p+q}$
2. $(A^p)^q = A^{pq}$

Remark:

The equality $(AB)^p = A^p B^p$ need not be true except when $AB = BA$. (**Why?**)

Upper and Lower Triangular Matrices

An $n \times n$ matrix $A = [a_{ij}]$ is called **upper triangular** if $a_{ij} = 0$ for $i > j$ and called **lower triangular** if $a_{ij} = 0$ for $i < j$.

EXAMPLE 15. Upper triangular matrix:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

Lower triangular matrix:

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 5 & 2 & 1 \end{bmatrix}$$

QUESTIONS:

1. Can upper triangular matrix be a lower triangular matrix? **ANSWER:** ________________
2. Is it possible for a lower triangular matrix be upper triangular? **ANSWER:** ________________
3. Is it possible for a matrix to be both upper triangular and lower triangular? **ANSWER:** ________________

Some other special types of matrices:

Symmetric and Skew Symmetric Matrices

A matrix A with real entries is called **symmetric** if $A^T = A$.
A matrix A with real entries is **skew symmetric** if $A^T = -A$.

EXAMPLE 16. Symmetric matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 3 & 3 & -1 \end{bmatrix}$$

Skew symmetric matrix:

$$B = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$$

Remarks:

1. If A is symmetric or skew symmetric, then A is a square matrix.
2. If A is a symmetric matrix, then the entries of A are symmetric with respect to the main diagonal of A.
3. A is symmetric if and only if $a_{ij} = a_{ji}$.
4. A is skew symmetric if and only if $a_{ij} = -a_{ji}$.
5. If A is skew symmetric, the entries of the main diagonal of A are all zero.

ASSIGNMENT:

Each item is worth 10 points. Use short bondpaper.

Prove the following:

1. Show that if A is a symmetric matrix, then A^T is symmetric.
2. Show that if A is any $m \times n$ matrix, then AA^T and A^TA are symmetric.
3. Show that if A is any $n \times n$ matrix, then $A + A^T$ is symmetric.
4. Show that if A is any $n \times n$ matrix, then $A - A^T$ skew symmetric.
5. Let A and B be symmetric matrices. Show that $A + B$ is symmetric.
6. Let A and B be symmetric matrices.Show that $A + B$ is symmetric if and only if $AB = BA$.

Chapter 3

Nonsingular Matrices

3.1 Nonsingular Matrices

Another special type of a matrix corresponds to the reciprocal of a nonzero real number.

Nonsingular Matrices

An $n \times n$ matrix A is called **nonsingular** or **invertible** if there exist an $n \times n$ matrix B such that $AB = BA = I_n$. This matrix B is called an inverse of A. Otherwise, A is called **singular** or **noninvertible**.

Theorem

THEOREM 5. The inverse of a matrix, if there is, is unique.

Proof: Suppose that B and C are inverses of a matrix A. Then by definition, $AB = BA = I_n$ and $AC = CA = I_n$. It follows that

$$\begin{aligned} B &= BI_n \\ &= B(AC) \\ &= (BA)C \\ &= I_nC \\ &= C \end{aligned}$$

This shows that the inverse of a matrix, if it exists, is unique. □

NOTATION: Because the inverse of a matrix is unique, we denote the inverse of A as A^{-1}.

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Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Compute A^{-1}.

Inverse of the Product, Inverse, and Transpose of Matrices

THEOREM 6.

1. If A and B are both nonsingular $n \times n$ matrices, then AB is nonsingular and
$$(AB)^{-1} = B^{-1}A^{-1}.$$
2. If $A_1, A_2, \ldots, A_r$ are $n \times n$ nonsingular matrices, then $A_1A_2\ldots A_r$ is nonsingular and
$$(A_1A_2\ldots A_r)^{-1} = A_r^{-1}A_{r-1}^{-1}\ldots A_1^{-1}.$$
3. If A is a nonsingular matrix, then A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.
4. If A is a nonsingular matrix, then A^T is nonsingular and $(A^{-1})^T = (A^T)^{-1}$.

Proof:

1 Note that
$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= (AI_n)A^{-1} \\ &= AA^{-1} \\ &= I_n.\end{aligned}$$
Similarly, $(B^{-1}A^{-1})(AB) = I_n$. This means that AB is nonsingular. By previous theorem, the inverse of a matrix is unique, so $(AB)^{-1} = B^{-1}A^{-1}$.

The proofs for $2, 3$ and 4 will be left as an exercise. ■

ASSIGNMENT:

Each item is worth 10 points. Use short bondpaper.

Prove the following:

1. Show that if $AB = AC$ and A is nonsingular, then $B = C$.
2. Show that if A is nonsingular and $AB = O$ for $n \times n$ matrix B, then $B = O$.
3. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that A is nonsingular if and only if $ad - bc \neq 0$.
4. Consider the homogeneous system $Ax = 0$, where A is $n \times n$. If A is nonsingular, show that the only solution is the trivial one, $x = 0$.
5. Prove that if A is symmetric and nonsingular, then A^{-1} is symmetric.

Chapter 4

Echelon Form of a Matrix

4.1 Echelon Form of a Matrix

This chpater introduces other operations on a matrix that will help in solving for a solution of a linear system

Consider this definition.

Reduced Row Echelon Form

An $m \times n$ matrix A is said to be in **reduced row echelon form** if it satisfies the following:

1. All zero rows, if it exists, appear at the bottom of the matrix.
2. The first nonzero entry from the left of a nonzero row is a 1. This is called the **leading one** of its row.
3. For each nonzero row, the leading one appears to the right and below any leading ones in the preceding rows.
4. If a column contains a leading one, then all other entries in that column are zero.

An $m \times n$ matrix satisfying only the properties $1, 2$, and 3 is said to be in **row echelon form**.

We can formulate similar definition for **reduced column echelon form** and **column echelon form**.

EXAMPLE 17. Given

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

QUESTION: Is A in reduced row echelon form? row echelon form? **ANSWER:** _______________

EXAMPLE 18. Given

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

QUESTION: Is B in reduced row echelon form? row echelon form? **ANSWER:** _______________

EXAMPLE 19. Given

$$C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

QUESTION: Is C in reduced row echelon form? row echelon form? **ANSWER:** _______________

EXAMPLE 20. Given

$$D = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

QUESTION: Is D in reduced row echelon form? row echelon form? **ANSWER:** _______________

EXAMPLE 21. Given

$$E = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 2 & -2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

QUESTION: Is E in reduced row echelon form? row echelon form? **ANSWER:** _______________

EXAMPLE 22. Given

$$F = \begin{bmatrix} 1 & 5 & 0 & 2 & -2 & 4 \\ 0 & 1 & 0 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

QUESTION: Is F in reduced row echelon form? row echelon form? **ANSWER:** _______________

QUESTIONS:

Are all matrices in its reduced row echelon form also in row echelon? **ANSWER:** _______________

All all matrices in row echelon form also in reduced row echelon form? **ANSWER:** _______________

We now show that every matrix can be transformed into row (column) echelon form or into reduced row (column) echelon form by row (column operations.

Elementary Row(Column) Operations

An **elementary row (column) operation** on a matrix A is any one of the following operations:

1. **Type I**: Interchange any teo rows (columns).
2. **Type II**: Multiply a row (column) by a nonzero constant.
3. **Type III**: Add a multiple of one row (column) to another.

NOTATIONS:

Type I: $r_i \leftrightarrow r_j$

Type II: $kr_i \to r_i$

Type III: $kr_i + r_j \to r_j$

Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & 9 \end{bmatrix}$$

Then

$$B = A_{r_1 \leftrightarrow r_3} = \begin{bmatrix} 3 & 3 & 6 & 9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

$$C = A_{\frac{1}{3}r_3 \to r_3} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{bmatrix}$$

and

$$D = A_{-2r_2 + r_3 \to r_3} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}$$

Row(Column) Equivalent

An $m \times n$ matrix B is said to be **row (column) equivalent** to an $m \times n$ matrix A if B can be produced by applying a finite sequence of elementary row (column) operations to A.

EXAMPLE 23. The matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & 9 \end{bmatrix}$$

and

$$D = A_{-2r_2+r_3\to r_3} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ -1 & -3 & 6 & -5 \end{bmatrix}$$

are row equivalent.

Row(Column) Equivalent

Every nonzero $m \times n$ matrix A is row (column) equivalent to a matrix in row (column) echelon form.

EXAMPLE 24. Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & 9 \end{bmatrix}.$$

Here 2 is called the pivot. (A **pivot column** is the first column with nonzero entry. A **pivot** is the first nonzero entry in the pivot column)

Then

$$B = A_{r_1\leftrightarrow r_2} = \begin{bmatrix} 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 3 & 3 & 6 & 9 \end{bmatrix}$$

$$C = B_{\frac{1}{2}r_1\to r_1} = \begin{bmatrix} 1 & 3/2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 3 & 3 & 6 & 9 \end{bmatrix}$$

$$D = C_{-3r_1+r_3\to r_3} = \begin{bmatrix} 1 & 3/2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & -3/2 & 6 & 11 \end{bmatrix}$$

$$E = D_{r_2\leftrightarrow r_3} = \begin{bmatrix} 1 & 3/2 & 0 & -1 \\ 0 & -3/2 & 6 & 11 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Then

$$F = E_{\frac{-2}{3}r_2\to r_2} = \begin{bmatrix} 1 & 3/2 & 0 & -1 \\ 0 & 1 & -4 & -22/3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Observe that F is now in row echelon form. We can continue to until we get a matrix in reduced row echelon form.

Theorem

THEOREM 7. Every nonzero $m \times n$ matrix A is row (column) equivalent to a unique matrix in reduced row (column) echelon form.

Remark:

The row echelon form of a matrix is not unique. (**ASSIGNMENT:** Provide an example of this)

NAME: ____

Find a row echelon form of the following matrix. Record the row operations you perform using the notations for elementary row operations.

$$A = \begin{bmatrix} -1 & 2 & -5 \\ 2 & -1 & 6 \\ 2 & 2 & 7 \end{bmatrix}$$

Chapter 5

Solving Linear Systems

5.1 Solving Linear Systems

In this chapter, we will use the echelon form of a matrix to determine the solution of a linear system.

Equivalent Linear Systems

THEOREM 8. Let $Ax = b$ and $Cx = d$ be two linear systems, each of m equations in n unknowns. If the augmented matrix $[A|b]$ and $[C|d]$ are row equivalent, then the linear systems are equivalent, i.e, they have exactly the same solutions.

Corollary

If A and C are row equivalent $m \times n$ matrices, then the homogeneous systems $Ax = 0$ and $Cx = 0$ are equivalent.

We have two methods for solving linear systems:

1. Gaussian elimination
2. Gauss-Jordan reduction

STEPS FOR GAUSSIAN ELIMINATION:

1. Transform the augmented matrix $[A|b]$ to the matrix $[C|d]$ in row echelon form using elementary row operations.

2. Write the solution of the linear system corresponding to the augmented matrix $[C|d]$ using back substitution.

We follow the same steps for Gauss-Jordan reduction replacing row echelon form by reduced row echelon form.

EXAMPLE 25. Find the solution of the linear system

$$\begin{aligned} x + 2y + 3z &= 9 \\ 2x - y + z &= 8 \\ 3x - z &= 3 \end{aligned}$$

Solution. The linear system

$$\begin{aligned} x + 2y + 3z &= 9 \\ 2x - y + z &= 8 \\ 3x - z &= 3 \end{aligned}$$

has the augmented matrix

$$[A|b] = \begin{bmatrix} 1 & 2 & 3 & | & 9 \\ 2 & -1 & 1 & | & 8 \\ 3 & 0 & -1 & | & 3 \end{bmatrix}$$

Transforming this matrix to row echelon form, we have

$$[C|d] = \begin{bmatrix} 1 & 2 & 3 & | & 9 \\ 0 & 1 & 1 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

Using back substitution, we get

$$\begin{aligned} z &= 3 \\ y &= 2 - z = 2 - 3 = -1 \\ x &= 9 - 2y - 3z = 9 + 2 - 9 = 2 \end{aligned}$$

Hence, the solution of the linear system is $x = 2, y = -1, z = 3$. □

EXAMPLE 26. Let

$$[C|d] = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & | & 6 \\ 0 & 1 & 2 & 3 & -1 & | & 7 \\ 0 & 0 & 1 & 2 & 3 & | & 7 \\ 0 & 0 & 0 & 1 & 2 & | & 9 \end{bmatrix}$$

Then

$$\begin{aligned}
x_4 &= 9 - 2x_5 \\
x_3 &= 7 - 2x_4 - 3x_5 = -11 + x_5 \\
x_2 &= 2 - 2x_3 = 2 + 5x_5 \\
x_1 &= 6 - 2x_2 - 3x_3 - 4x_4 - 5x_5 = -1 - 10x_5 \\
x_5 &= \text{ any real number.}
\end{aligned}$$

The system is consistent, and all solutions are of the form

$$\begin{aligned}
x_1 &= -1 - 10r \\
x_2 &= 2 + 5r \\
x_3 &= -11 + r \\
x_4 &= 9 - 2r \\
x_5 &= r, \text{ any real number.}
\end{aligned}$$

The given linear system has infinitely many solutions.

EXAMPLE 27. If

$$[C|d] = \begin{bmatrix} 1 & 2 & 3 & | & 9 \\ 0 & 1 & 1 & | & 2 \\ 0 & 0 & 0 & | & 3 \end{bmatrix}$$

then the linear system $Cx = d$ has no solution since the last equation $0x_1 + 0x_2 + 0x_3 = 3$ is not true.

EXAMPLE 28. If

$$[C|d] = \begin{bmatrix} 1 & 0 & 0 & 0 & | & 9 \\ 0 & 1 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & 0 & | & 3 \\ 0 & 0 & 0 & 1 & | & 2 \end{bmatrix}$$

then the solution of the linear system is $x_1 = 9, x_2 = 2, x_3 = 3, x_4 = 2$.

This procedure is the Gauss-Jordan reduction.

EXAMPLE 29. Consider the linear system

$$\begin{aligned}
x + 2y + 3z &= 9 \\
2x - y + 2z &= 14 \\
3x + y - z &= -2.
\end{aligned}$$

Its augmented matrix is

$$[A|b] = \begin{bmatrix} 1 & 2 & 3 & | & 6 \\ 2 & -3 & 2 & | & 14 \\ 3 & 1 & -1 & | & -2 \end{bmatrix}$$

Converting this to its row echelon form, we get

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 14 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Thus, $z = 3, y = -2, x = 1$ by back substitution. This procedure is by Gaussian elimination.

To solve the linear system by Gauss-Jordan reduction, we transfrom the matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 14 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

to its reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

We can see that this has the solution $z = 3, y = -2, x = 1$ is the same by using Gaussian elimination procedure.

NAME: ____

Consider the linear system

$$\begin{aligned} x + y + 2z &= -1 \\ x - 2y + 2z &= -5 \\ 3x + y + z &= 3. \end{aligned}$$

1. Find all solutions, if any exists, by using the Gaussian elimination method.
2. Find all solutions, if any exists, by using the Gauss-Jordan reduction method.

Chapter 6

Inverse of a Matrix

6.1 Inverse of a Matrix

In this lesson, we will find for the inverse of a matrix by using elementary row operations. Consider first the following definition:

Definition

An $n \times n$ **elementary matrix of type I, type II, or type III** is a matrix obtained from the identity matrix I_n by performing a single elementary row or elementary column operation of type I, type II, or type III, respectively.

EXAMPLE 30.

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

E_1 is of type I (we interchanged the first and third rows of I_3)

EXAMPLE 31.

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E_2 is of type II (we multiplied the second row of I_3 by -3)

EXAMPLE 32.

$$E_3 = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E_3 is of type III (we added four times the second row of I_3 to the first row of I_3)

EXAMPLE 33.

$$E_4 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E_4 is of type III (we added two times the first column of I_3 to the third column of I_3)

An elementary row operation on a matrix A can be achieved by premultiplying A (multiplying A on the left) by corresponding elementary matrix E.

Also, an elementary column operation on A can be obtained by postmultiplying A (multiplying A on the right) by corresponding elementary matrix.

Formally, we have the following theorem:

Performing Elementary Row(Column) Operation

THEOREM 9. Let A be an $m \times n$ matrix, and let an elementary row(column) operation of type I, type II, or type III be performed on A to yield matrix B. Let E be the elementary matrix obtained from I_m (I_n) by performing the same elementary row(column) operation as was performed on A. Then $B = EA$ ($B = AE$).

EXAMPLE 34. Let

$$A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ -1 & 2 & 3 & 4 \\ 3 & 0 & 1 & 2 \end{bmatrix}$$

and let $B = A_{-2r_3+r_1 \to r_1}$. Then

$$B = \begin{bmatrix} -5 & 3 & 0 & -3 \\ -1 & 2 & 3 & 4 \\ 3 & 0 & 1 & 2 \end{bmatrix}.$$

Let $E = (I_3)_{-2r_3+r_1 \to r_1}$. Then

$$E = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can check that $B = EA$.

Some important results:

Row(Column) Equivalent

THEOREM 10. If A and B are $m \times n$ matrices, then A is row(column) equivalent to B if and only if there exist elementary matrices $E_1, E_2, \ldots, E_k$ such that $B = E_k E_{k-1} \cdots E_2 E_1 A \quad (B = AE_1 E_2 \cdots E_{k-1} E_k)$.

An Elementary Matrix and its Inverse is Nonsingular

THEOREM 11. An elementary matrix E is nonsingular, and its inverse is an elementary matrix of the same type.

Reduced Row Echelon Form of a Coefficient Matrix

THEOREM 12. Let A be an $n \times n$ matrix and let the homogeneous system $Ax = 0$ have only the trivial solution $x = 0$. Then A is row equivalent to I_n, i.e, the reduced row echelon form of A is I_n.

Characterization

THEOREM 13. The following statements are equivalent for an $n \times n$ matrix A:

1. A is nonsingular
2. $Ax = 0$ has only the trivial solution
3. A is row(column) equivalent to I_n
4. The linear system $Ax = b$ has unique solution for every $n \times 1$ matrix b
5. A is a product of elementary matrices

Algorithm for finding A^{-1}:

1. Perform elementary row operations on A until we get I_n
2. The product of the elementary matrices $E_k E_{k-1} \cdots E_2 E_1$ gives A^{-1}

For convenience, we write down the partitioned matrix $[A|I_n]$ then transform to its reduced row echelon form to obtain $[I_n|A^{-1}]$.

EXAMPLE 35. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{bmatrix}.$$

If A is nonsingular, we form

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$$

Transforming $[A|I_3]$ to $[I_3|A^{-1}]$, we have

	A			I_3		
1	1	1	1	0	0	Apply $-5r_1 + r_3 \to r_3$
0	2	3	0	1	0	
5	5	1	0	0	1	
1	1	1	1	0	0	Apply $\frac{1}{2}r_2 \to r_2$
0	2	3	0	1	0	
0	0	-4	-5	0	1	
1	1	1	1	0	0	Apply $-\frac{1}{4}r_3 \to r_3$
0	1	$\frac{3}{2}$	0	$\frac{1}{2}$	0	
0	0	-4	-5	0	1	
1	1	1	1	0	0	Apply $\frac{-3}{2}r_3 + r_2 \to r_2$ and $-1r_3 + r_1 \to r_1$
0	1	$\frac{3}{2}$	0	$\frac{1}{2}$	0	
0	0	1	$\frac{5}{4}$	0	$-\frac{1}{4}$	
1	1	0	$-\frac{1}{4}$	0	$\frac{1}{4}$	Apply $-1r_2 + r_1 \to r_1$
0	1	0	$-\frac{15}{8}$	$\frac{1}{2}$	$\frac{3}{8}$	
0	0	1	$\frac{5}{4}$	0	$-\frac{1}{4}$	
1	0	0	$\frac{13}{8}$	$-\frac{1}{2}$	$-\frac{1}{8}$	
0	1	0	$-\frac{15}{8}$	$\frac{1}{2}$	$\frac{3}{8}$	
0	0	1	$\frac{5}{4}$	0	$-\frac{1}{4}$	

It follows that

$$A^{-1} = \begin{bmatrix} \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{bmatrix}.$$

QUESTION: How do we know that a matrix A is singular?

The next theorem answers this question.

Singular Matrix

THEOREM 14. An $n \times n$ matrix A is singular if and only if A is row equivalent to matrix B, having at least one row that consists entirely of zeros.

EXAMPLE 36. Let

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ 5 & -2 & -3 \end{bmatrix}.$$

To find A^{-1}, we proceed as follows:

A				I_3		
1	2	−3	1	0	0	Apply $-1r_1 + r_2 \to r_2$
1	−2	1	0	1	0	
5	−2	−3	0	0	1	
1	2	−3	1	0	0	Apply $-5r_1 + r_3 \to r_3$
0	−4	4	−1	1	0	
5	−2	−3	0	0	1	
1	2	−3	1	0	0	Apply $-3r_2 + r_3 \to r_3$
0	−4	4	−1	1	0	
0	−12	12	−5	0	1	
1	2	−3	1	0	0	
0	−4	4	−1	1	0	
0	0	0	−2	−3	1	

It follows that A is row equivalent to

$$B = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since B has row of zeros, we conclude that A is a singular matrix.

NAME: ____

1. Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 4 \\ 3 & 0 & 1 \end{bmatrix}$$

if it exists.

Chapter 7

Determinants

Consider first the following definition:

> **Definition**
>
> Let $S = \{1, 2, \ldots, n\}$ be the set of integers from 1 to n arranged in ascending order. A permutation of S is a rearrangement $j_1 j_2 \ldots j_n$ of elements of S. We can view a permutation of S as a one-to-one mapping of S onto itself.

EXAMPLE 37. Let $S = \{1, 2, 3, 4\}$. Then 4231 is a permutation of S. This corresponds to the function $f : S \to S$ defined by $f(1) = 4, f(2) = 2, f(3) = 3, f(4) = 1$.

> **Definition**
>
> A permutation $j_1 j_2 \ldots j_n$ is said to have an **inversion** if a larger integer j_r precedes a smaller one j_s. A permutation is called **even** if the total number of inversions in it is even or **odd** if the total number of inversions in it is odd.

Remark: If $n \geq 2$, there are $n!/2$ even and $n!$ odd permutations in S_n

EXAMPLE 38.

1. S_1 has only $1! = 1$ permutation, which is even because there is no inversion.
2. S_2 has $2!$ permutations:
   12, which is even (no inversion)
   21, which is odd (one inversion)
3. In the permutation 4321 of S_4, the total number of inversions is 5, hence it is odd.

Definition

Let $A = [a_{ij}]$ be $n \times n$ matrix. The **determinant** function, denoted by **det**, is defined by

$$det(A) = \sum (\pm) a_{1j_1} a_{2j_2} \cdots a_{nj_n},$$

where the summation is over all permutations $j_1 j_2 \cdots j_n$ is even or odd.

The sign is $+$ if the permutation is even and $-$ if the permutations is odd.

Other notation: $det(A) = |A|$

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

To obtain $det(A)$, write the terms $a_{1\_}a_{2\_}$ and replace the dashes with all possible elements of S_2.

The subscripts become 12 and 21.

Since 12 is even and 21 is an odd permutation,

$$det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Practice:

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 0 & 1 \end{bmatrix}$$

Evaluate $det(A)$.

NAME: ____

Given a 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

determine $det(A)$ by using the definition.

The previous method for computing $det(A)$ is not applicable for $n \geq 4$.

PROPERTIES OF DETERMINANTS:

1. If A is a matrix, then $det(A) = det(A^T)$.
2. If matrix B results from matrix A by interchanging two different rows (columns) of A, then $det(B) = -det(A)$.
3. If a row (column) of A consists of entirely zeros, then $det(A) = 0$.
4. If B is obtained from A by multiplying a row (column) of A by a real number k, then $det(B) = kdet(A)$.
5. If $B = [b_{ij}]$ is obtained from $A = [a_{ij}]$ by adding to each element of the rth row (column) of A, k times the corresponding element of the sth row (column), $r \neq s$, of A, then $det(B) = det(A)$.

   EXAMPLE 39.
   $$\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 9 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix}$$

   How?

6. If a matrix $A = [a_{ij}]$ is upper (lower) triangular, then $det(A) = a_{11}a_{22}\cdots a_{nn}$; that is, the determinant of a triangular matrix is the product of the elements on the main diagonal.
7. If A is an $n \times n$ matrix, then A is nonsingular if and only if $det(A) \neq 0$.
8. If A and B are $n \times n$ matrices, then $det(AB) = det(A)det(B)$.
9. If A is nonsingular, then $det(A^{-1}) = \frac{1}{det(A)}$.

We will now develop a method for evaluating the determinant of an $n \times n$ matrix that reduces the problem to the evaluation of matrices of order $n - 1$.

Definition

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Let M_{ij} be the $(n-1) \times (n-1)$ submatrix of A obtained by deleting the ith row and jth column of A. The determinant $det(M_{ij})$ is called the **minor** of a_{ij}.
Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **cofactor** A_{ij} of a_{ij} is defined as $A_{ij} = (-1)^{i+j}det(M_{ij})$.

Practice:

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{bmatrix}.$$

Evaluate $det(M_{12})$, $det(M_{23})$, and $det(M_{31})$, A_{12}, A_{23}, and A_{31}

Remark:

To evaluate the determinant of a matrix, it is best to expand along either row or column with largest number of zeroes. (**Why?**)

7.1 Cramer's Rule

Let

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad &\quad\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

be a linear system of n equations in n unknowns and let $A = [a_{ij}]$ be the coefficient matrix so that we can write the given system as $A\vec{x} = \vec{b}$, where

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

If $det(A) \neq 0$, then the system has the unique solution

$$x_1 = \det \frac{det(A_1)}{det(A)}, x_2 = \det \frac{det(A_2)}{det(A)}, \cdots, x_n = \det \frac{det(A_n)}{det(A)},$$

where A_i is the matrix obtained from A by replacing the ith column of A by $\vec{b}$.

EXAMPLE 40. Consider the following linear system:

$$\begin{aligned} -2x_1 + 3x_2 - 3x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= 4 \\ -2x_1 - x_2 + x_3 = -3 \end{aligned}$$

Solution.

$$A = \begin{bmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix} = -2$$

Then

$$x_1 = \frac{\begin{bmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{bmatrix}}{|A|} = \frac{-4}{-2} = 2,$$

$$x_2 = \frac{\begin{bmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{bmatrix}}{|A|} = \frac{-6}{-2} = 3,$$

and

$$x_3 = \frac{\begin{bmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{bmatrix}}{|A|} = \frac{-8}{-2} = 4.$$

□

NAME: ____

1. Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -4 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

by using the definition and cofactor method.

2. If possible, solve the following linear systems by Cramer's rule:

$$\begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 2 \\ x_1 + 2x_3 &= 0 \\ 2x_1 + 3x_2 - x_3 &= -5 \end{aligned}$$

NAME: ____

Write **TRUE** if the following statements are correct and **FALSE** if it is not correct.

1. $det(A + B) = det(A) + det(B)$ **ANSWER:** __________
2. $det(A^{-1}B) = \dfrac{det(B)}{det(A)}$ **ANSWER:** __________
3. If $det(A) = 0$, then A has atleast two equal rows. **ANSWER:** __________
4. If A has a column of all zeros, then $det(A) = 0$. **ANSWER:** __________
5. A is singular if and only if $det(A) = 0$. **ANSWER:** __________
6. If B is the reduced row echelon form of A, then $det(B) = det(A)$. **ANSWER:** __________
7. The determinant of an elementary matrix is always equal to 1. **ANSWER:** __________
8. If all the diagonal elements of an $n \times n$ matrix A are zero, then $det(A) = 0$. **ANSWER:** __________
9. $det(AB^TA^{-1}) = det(B)$. **ANSWER:** __________
10. $\frac{1}{c}(det(cA) = det(A)$. **ANSWER:** __________

Chapter 8

Vectors in Plane and 3-Space

8.1 Vectors in Plane and in 3-Space

In many applications we deal with measurable quantities such as pressure, mass, and speed. This can be described using their magnitude. They are called **scalars**.

Other measurable quantities such as velocity, force, and acceleration require both magnitude and direction. They are called **vectors**.

This chapter introduces vectors in a plane and 3-space.

We note that vectors may be encountered in Physics and in Calculus.

NOTATIONS:

$\vec{v}$ means vector $\vec{v}$

Small letters of English alphabet for scalars

ACTIVITY:

In a short bondpaper, perform the following:

1. Draw a pair of perpendicular lines intersecting at a point O. Name this as the **origin**. Name the horizontal line as x-axis and the vertical line as the y-axis. Together it is called a **coordinate axes** and forms a **rectangular coordinate system/Cartesian coordinate system**.

2. Choose a point on the x-axis to the right of O and a point on the y-axis above O.

3. Associate an ordered pair (x, y) to the assigned points. We call x and y as coordinates of (x, y). This point P in the Cartesian coordinate system can be denoted as $P(x, y)$.

   The set of all points in the plane is denoted by $\mathbb{R}^2$. This is also called the **2-space**.

4. Associate a 2×1 matrix corresponding to $P(x, y)$. Write this as

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

5. Draw a directed line segment from O to $P(x, y)$. Here, O is called its tail and P its head. (To distinguish the tail from head, draw an arrowhead on the head)

   Notice that this directed line segment has a **direction**. The **magnitude** of a directed line segment is its length.

   Can you compute the magnitude of your directed line segment?

6. Define based on your experience from 1 to 5 a vector in the plane.

Definition

A **vector in the plane** is a 2×1 matrix $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, where x and y are real numbers, called the **components** of $\vec{x}$. We call a vector in a plane as a **vector** or a **2-vector**.

QUESTIONS:

1. With a given vector $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, can we associate a directed line segment?

   ANSWER: __________

2. Given a directed line segment, can we associate a vector?

   ANSWER: __________

3. Can we use the terms directed line segment and vector interchangeably?

   ANSWER: __________

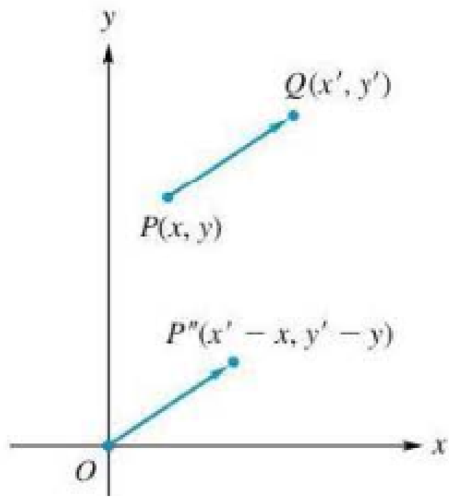
In physical conditions, it is necessary to deal with a directed line segment PQ from point $P(x, y)$ not necessarily the origin to the point $Q(x', y')$. This is also called a vector in a plane with tail P and head Q.

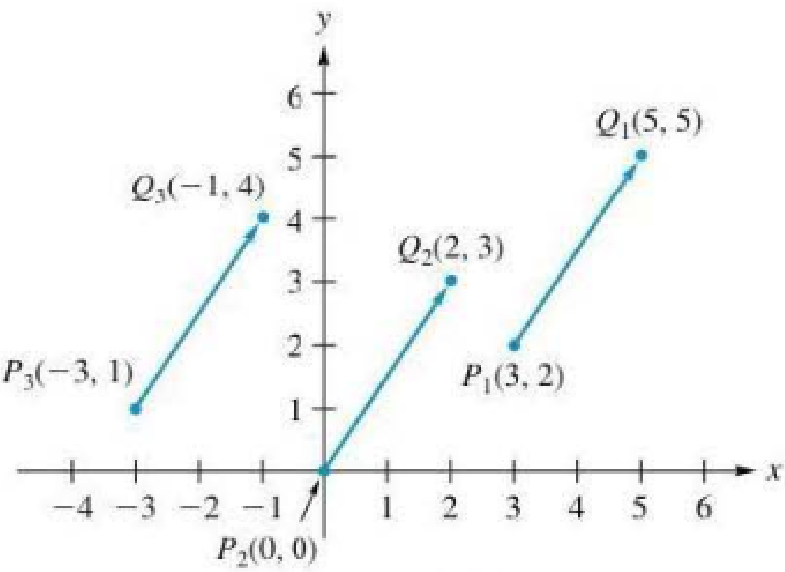
What do you think is the component of this vector?

ANSWER: __________

Definition

Two vectors in the plane are **equal** if their respective components are equal.





Are the following vectors are equal? What is the component of the three vectors?

EXAMPLE 41. The head of the vector

$$\vec{P_4Q_4} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \vec{P_2Q_2}$$

with tail $P_4(-5, 2)$ can be determined as follows:

$$x_4' - (-5) = 2$$

and $y_4' - 2 = 3$

Thus, $x_4' = -3$ and $y_4' = 5$.

QUESTION:

What is the tail of the vector $P_5(x_5, y_5)$ of the vector

$$P_5\vec{Q}_5 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

with head $Q_5(8, 6)$?

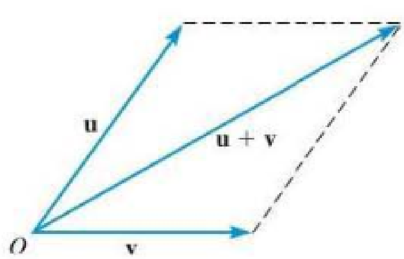
Note that a vector in a plane can represented as a matrix. Hence, definition for operations like addition and scalar multiplication applies.

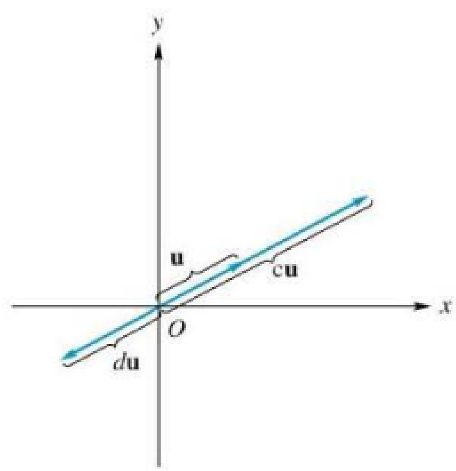
Definition

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be two vectors in the plane. The sum of the vectors $\vec{u}$ and $\vec{v}$ is the vector $u \vec{+} v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$

If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a vector and c is a scalar (real number), then the scalar multiple $c\vec{u}$ of $\vec{u}$ by c is the vector $c\vec{v} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}$.

Geometrically, vector addition and scalar multiplication can be represented as follows:



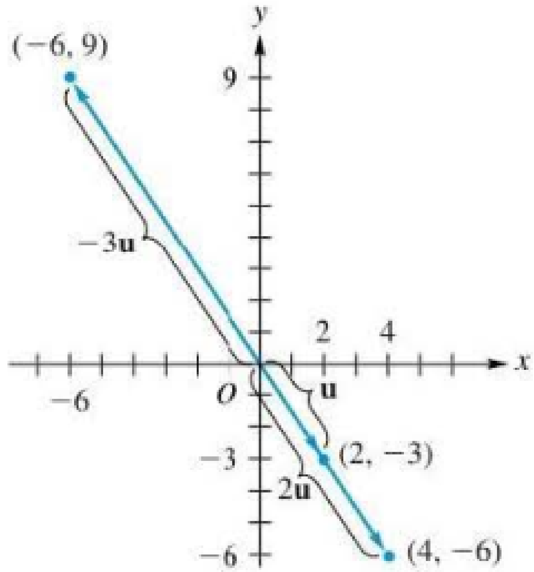


Here, $c > 0$ and $d < 0$.

EXAMPLE 42. If $c = 2, d = -3$, and $\vec{u} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

The vector $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is called the **zero vector**.

RESULTS:

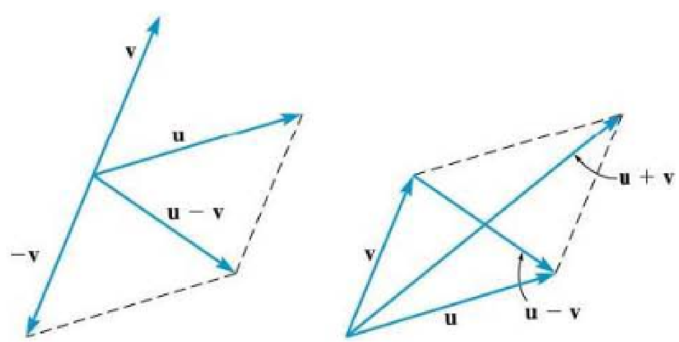


1. If $\vec{u}$ is any vector, then $\vec{u}+\vec{0}=\vec{u}$.
2. For any vector $\vec{u}$, $\vec{u}+(-1)\vec{u}=\vec{0}$.

We write $(-1)\vec{u}$ as $-\vec{u}$. This is called the **negative** of $\vec{u}$.

NOTATION: $\vec{u}+(-1)\vec{u}=u\vec{-}v$.

Difference between two vectors can be represented geometrically as follows:



NAME: ____

Sketch a directed line segment in $\mathbb{R}^2$ representing the following:

1. $\vec{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

2. $\vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

3. $u \vec{+} v$ (Use vectors in 1 and 2)

4. $u \vec{-} v$ (Use vectors in 1 and 2)

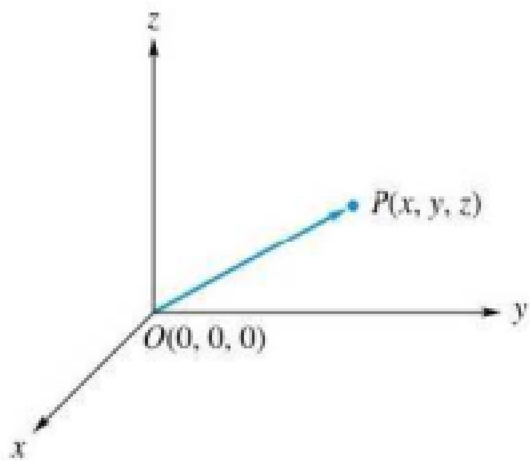
5. Determine the head of the vector $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ whose tail is $(-3, 2)$. Sketch this vector.

Equivalently, we can associate a point $P(x, y, z)$ in $\mathbb{R}^3$ a vector.

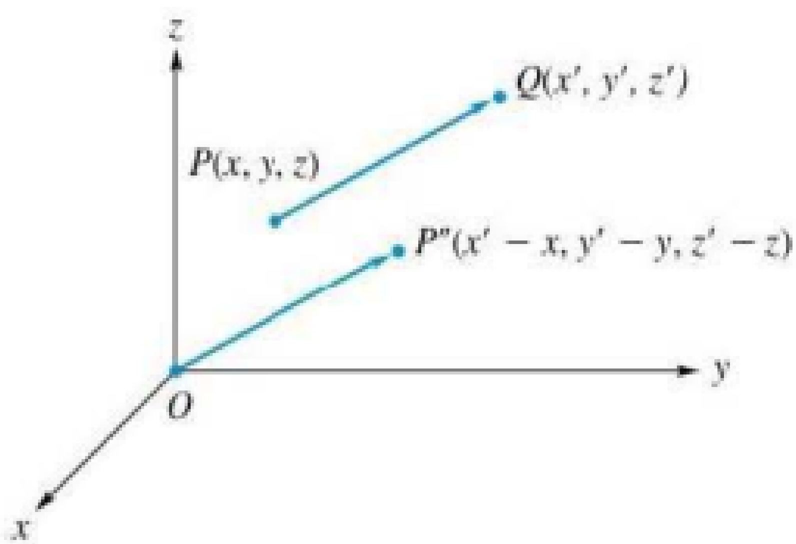
Definition

A **vector in space**, or **3-vector**, or simply a **vector**, is a 3×1 matrix $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, where x, y, z are real numbers called the components of $\vec{x}$.
Two vectors in space are said to be **equal** if their respective components are equal.

Geometrically, a vector in $\mathbb{R}^3$ can be represented as



Similar with vectors in a plane, the following represent different directed line segments representing the same vector.



Vector addition and scalar multiplication in $\mathbb{R}^3$ are defined in similar way.

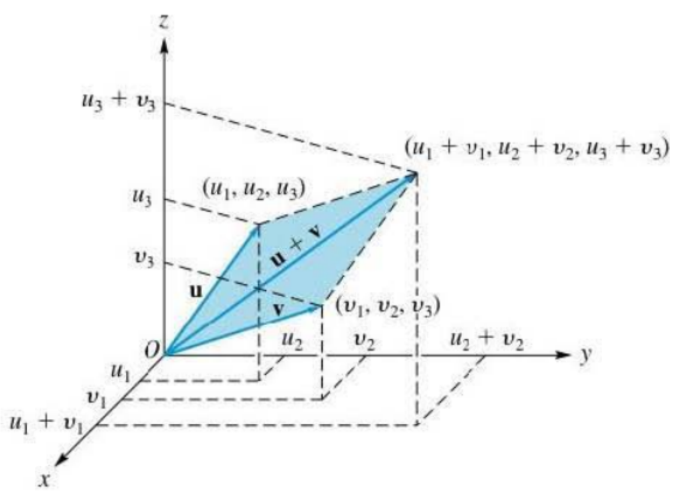
Definition

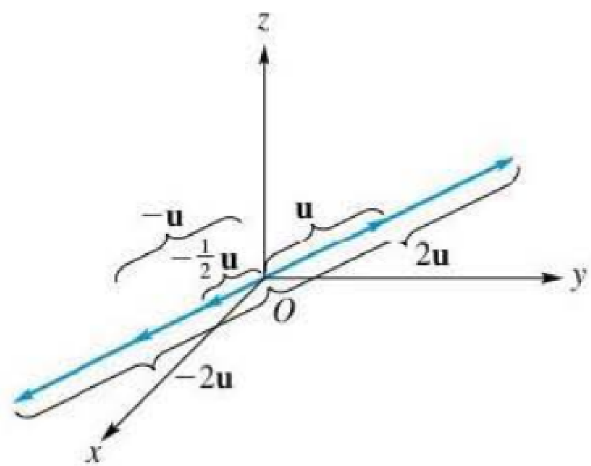
Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be two vectors in $\mathbb{R}^3$. The sum of the vectors $\vec{u}$ and $\vec{v}$ is the vector $u \vec{+} v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$

If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ is a vector and c is a scalar (real number), then the scalar multiple $c\vec{u}$ of $\vec{u}$ by c is the vector $c\vec{v} = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}$.

What do you think is the zero vector and difference of two vectors mean in $\mathbb{R}^3$?

Geometrically,





NAME: ______________________________

Sketch a directed line segment in $\mathbb{R}^3$ representing the following:

1. $\vec{u} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$
2. $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$
3. $u \vec{+} v$ (Use vectors in 1 and 2)
4. Determine the head of the vector $\begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$ whose tail is $(3, 2, 1)$. Sketch this vector.

Chapter 9

Vector Spaces

9.1 Vector Spaces

PROPERTIES OF VECTORS in $\mathbb{R}^2$ or $\mathbb{R}^3$:

Properties of Vectors

THEOREM 15. If $\vec{u}, \vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, and c, d are real scalars, then

1. $\vec{u} + \vec{v} = u \vec{+} v$
2. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$
3. $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$
4. $\vec{u} + (-\vec{u}) = \vec{0}$
5. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
6. $(c + u)\vec{v} = c\vec{u} + d\vec{u}$
7. $c(d\vec{u}) = (cd)\vec{u}$
8. $1\vec{u} = \vec{u}$

We will now define a real vector space.

Definition

A **real vector space** is a set V of elements with two operations $\oplus$ and $\odot$ satisfying the following:

a If $\vec{u}$ and $\vec{v}$ are members of V, then $\vec{u} \oplus \vec{v} \in V$.

1 $\vec{u} \oplus \vec{v} = \vec{v} \oplus \vec{u}$ for each $\vec{u}, \vec{v} \in V$

2 $\vec{u} \oplus (\vec{v} \oplus \vec{w}) = (\vec{u} \oplus \vec{v}) \oplus \vec{w}$ for all $\vec{u}, \vec{v}, \vec{w} \in V$.

3 There is an element $\vec{0} \in V$ s.t. $\vec{u} \oplus \vec{0} = \vec{u}$ for all $\vec{u} \in V$.

4 For each $\vec{u} \in V$, there is an element $-\vec{u} \in V$ s.t. $u \oplus -\vec{u} = -\vec{u} \oplus \vec{u} = \vec{0}$.

b If $\vec{u} \in V$ and $c \in \mathbb{R}$, then $c \odot \vec{u} \in V$.

5 $c \odot (\vec{u} \oplus \vec{v}) = c \odot \vec{u} \oplus c \odot \vec{v}$ for each $\vec{u}, \vec{v} \in V$ and $c \in \mathbb{R}$.

6 $(c + d) \odot \vec{u} = c \odot \vec{u} \oplus d \odot \vec{u}$ for any $\vec{u} \in V$ and any real numbers c and d.

7 $c \odot (d \odot \vec{u}) = (cd) \odot \vec{u}$ for any $\vec{u} \in V$ and any real numbers c and d.

8 $1 \odot \vec{u} = \vec{u}$ for any $\vec{u} \in V$.

The elements of V are called **vectors**: the elements of the set of real numbers $\mathbb{R}$ are called **scalars**.

The operation $\oplus$ is called **vector addition** : the operation $\odot$ is called **scalar multiplication**.

The vector $\vec{0}$ in property a3 is called a **zero vector**,

The vector $-\vec{u}$ in property a4 is called a **negative** of $\vec{u}$.

A set V and two operations $\oplus$ and $\odot$ satisfying all the properties of the definition is called a **vector space**.

NOTATION: $\langle V, \oplus, \odot \rangle$ is a vector space together with operations $\oplus$ and $\odot$

REMARK: The $\vec{0}$ and $-\vec{u}$ are unique.

Proof: Let 0_1 and 0_2 be zero vectors. Then

$$\vec{0}_1 \oplus \vec{0}_2 = \vec{0}_1$$

and

$$\vec{0}_1 \oplus \vec{0}_2 = \vec{0}_2.$$

Thus,

$$\vec{0}_1 = \vec{0}_1 \oplus \vec{0}_2 = \vec{0}_2.$$

This shows that $\vec{0}$ is unique.

CLAIM: $-\vec{u}$ is unique.

Proof: Let u_1 and u_2 be negatives of $\vec{u}$. Then

$$\vec{u} \oplus \vec{u}_1 = \vec{0}$$

and

$$\vec{u} \oplus \vec{u}_2 = \vec{0}.$$

Thus,

$$\vec{u_1} \oplus (\vec{u} \oplus \vec{u_1}) = \vec{u_1} \oplus (\vec{u} \oplus \vec{u_2}).$$

It follows that

$$(\vec{u_1} \oplus \vec{u}) \oplus \vec{u_1} = (\vec{u_1} \oplus \vec{u}) \oplus \vec{u_2}.$$

Hence,

$$\vec{0} \oplus \vec{u_1} = \vec{0} \oplus \vec{u_2}.$$

Therefore,

$$\vec{u_1} = \vec{u_2}.$$

Consequently, $-\vec{u}$ is unique. ■

EXAMPLE 43. Consider $\mathbb{R}^n$. This is the set of all $n \times 1$ matrices

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

with real entries.

Let $\oplus$ be matrix addition and $\odot$ be multiplication of a matrix by a real number (scalar multiplication).

It can be verified that $\langle \mathbb{R}^n, \oplus, \odot \rangle$ is a vector space.

EXAMPLE 44.

The set V of all $m \times n$ matrices with $\oplus$ as matrix addition and $\odot$ as multiplication of a matrix by real number is a vector space.

We denote this vector space as M_{mn}.

EXAMPLE 45.

The set V of all real numbers with $\oplus$ as the usual addition of real numbers and $\odot$ as the usual multiplication of real numbers is a vector space.

In this case, the members of the set of real numbers is both vectors and scalars.

EXAMPLE 46. Let $\mathbb{R}_n$ be the set of all $1 \times n$ matrices $[a_1 a_2 \ldots a_n]$, where $\oplus$ is defined as

$$[a_1 a_2 \ldots a_n] \oplus [b_1 b_2 \ldots b_n] = [a_1 + b_1 a_2 + b_2 \ldots a_n + b_n]$$

and $\odot$ is defined as

$$c \odot [a_1 a_2 \ldots a_n] = [ca_1 ca_2 \ldots ca_n].$$

Then $\langle \mathbb{R}_n, \oplus, \odot \rangle$ is a vector space.

If $A = [a_{ij}]$ is an $n \times n$ matrix, then the **trace** of A, denoted by $Tr(A)$, is defined as the sum of all elements on the main diagonal of A.

$$Tr(A) = \sum_{i=1}^{n} a_{ii}$$

PROPERTIES OF THE TRACE OF A MATRIX:

1. $Tr(cA) = cTr(A)$, where c is a real numebr
2. $Tr(A + B) = Tr(A) + Tr(B)$
3. $Tr(AB) = Tr(BA)$
4. $Tr(A^T) = Tr(A)$
5. $Tr(A^T A) \geq 0$

EXAMPLE 47. Let V be the set of all 2×2 matrices with trace equal to zero, that is,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$$

if $Tr(A) = a + d = 0$.

The operation $\oplus$ and $\odot$ are standard matrix addition and standard scalar multiplication, respectively.

It can be shown that $\langle V, \oplus, \odot \rangle$ is a vector space.

EXAMPLE 48. Recall: A **polynomial** in t is a function of the form $p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$, where $a_i' s$ are real numbers for all $i = 1, 2, \ldots, n$ and n is a nonnegative integer. If $a_n \neq 0$, then $p(t)$ is said to have **degree** n. The zero polynomial has no degree.

Let P_n be the set of all polynomials of degree less than or equal to n together with the zero polynomial.

Let $p(t), q(t) \in P_n$. Then

$$p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$$

and

$$q(t) = b_n t^n + b_{n-1} t^{n-1} + \cdots + b_1 t + b_0.$$

Define $p(t) \oplus q(t)$ as

$$p(t) \oplus q(t) = (a_n + b_n) t^n + (a_{n-1} + b_{n-1}) t^{n-1} + \cdots + (a_1 + b_1) t + (a_0 + b_0).$$

If c is a scalar, define $c \odot p(t)$ as

$$c \odot p(t) = (c a_n) t^n + (c a_{n-1}) t^{n-1} + \cdots + (c a_1) t + (c a_0).$$

It can be shown that P_n is a vector space.

EXAMPLE 49. Let V be the set of all real-valued functions defined on $\mathbb{R}$. If $f, g \in V$, define

$$(f \oplus g)(t) = f(t) + g(t).$$

If $f \in V$ and $c \in \mathbb{R}$, define $(c \odot f)(t) = cf(t)$. Then V is a vector space.

EXAMPLE 50. Let V be the set of all real multiples of exponential functions of the form e^{kx}, where k is any real number. Define $\oplus$ as

$$c_1 e^{kx} \oplus c_2 e^{mx} = c_1 c_2 e^{(k+m)x}$$

and scalar multiplication $\oplus$

$$r \oplus c_1 e^{kx} = r c_1 e^{kx}.$$

$\langle V, \oplus, \odot \rangle$ is **not** a vector space.

To see this, note that $e^{0x} = 1$ and for any vector $c_1 e^{kx} \in V$,

$$c_1 e^{kx} \oplus 1 = 1 \oplus c_1 e^{kx} = c_1 e^{kx}.$$

This means that 1 is the zero vector in V.

Note also that $0e^{kx} = 0 \in V$ but there is no vector $\vec{v} = c_1 e^{kx}$ in V such that $\vec{v} \oplus 0 = 1$.

EXAMPLE 51. Let V be the set of all real numbers with operations $\vec{u} \oplus \vec{v} = \vec{u} - \vec{v}$ (this means that the operation $\oplus$ is the usual subtraction) and $c \odot \vec{u}$ (this means that the operation $\odot$ is the usual multiplication).

Investigate if $\langle V, \oplus, \odot \rangle$ is a vector space or not.

EXAMPLE 52. Let V be the set of all ordered triples of real numbers (x, y, z) with operations

$$(a, b, c) \oplus (x, y, z) = (x, b + y, c + z)$$

and

$$c \odot (x, y, z) = (cx, cy, cz).$$

Investigate if $\langle V, \oplus, \odot \rangle$ is a vector space or not.

EXAMPLE 53. Let V be the set of all integers. Define $\oplus$ as the ordinary addition and $\oplus$ as the ordinary multiplication.

Investigate if $\langle V, \oplus, \odot \rangle$ is a vector space or not.

PROPERTIES COMMON TO ALL VECTOR SPACES

THEOREM 16. If V is a vector space, $\vec{u} \in V$ and c is any real number , then

1. $\vec{0} \odot \vec{u} = \vec{0}$
2. $c \odot \vec{0} = \vec{0}$
3. If $c \odot \vec{u} = \vec{0}$, then either $c = 0$ or $\vec{u} = 0$.
4. $(-1) \odot \vec{u} = -\vec{u}$

Proof:

1. $\vec{0} \odot \vec{u} = (\vec{0} \oplus \vec{0}) \odot \vec{u} = \vec{0} \odot \vec{u} \oplus \vec{0} \odot \vec{u}$. This means that $\vec{0} \odot \vec{u}$ is a zero in V. Since a zero in a vector space is unique, $\vec{0} \odot \vec{u} = \vec{0}$.
2. $c \odot \vec{0} = c \odot (\vec{0} \oplus \vec{0}) = c \odot \vec{0} \oplus c \odot \vec{0}$. Thus, $c \odot \vec{0} = \vec{0}$.
3. Let $c \odot \vec{u} = \vec{0}$. If $c = 0$, then we are done. Suppose that $c \neq 0$. Then $\frac{1}{c} \odot (c \odot \vec{u}) = \frac{1}{c} \odot \vec{0} = \vec{0}$.

   Also, $\frac{1}{c} \odot (c \odot \vec{u}) = [\frac{1}{c}(c)] \odot \vec{u} = 1 \odot \vec{u} = \vec{u}$.
   Consequently, $\vec{u} = \vec{0}$.
4. Note that $(-1) \odot \vec{u} \oplus \vec{u} = (-1) \odot \vec{u} \oplus 1 \odot \vec{u} = (-1+1) \odot \vec{u} = \vec{0} \odot \vec{u} = \vec{0}$. Since the inverse of a vector is unique, $(-1) \odot \vec{u} = -\vec{u}$.

Chapter 10

Subspaces

10.1 Subspaces

Let V be a vector space and W a nonempty subset of V. If W is a vector space with respect to the operations in V, then W is calleda **subspace** of V.

To check if a subset of a vector space is a vector space, we use the following result.

NOTATION: We denote $W \leq V$ if W is a subspace of V.

Subspace Criterion

Let $\langle V, \oplus, \odot \rangle$ be a vector space and $\varnothing \neq W \subseteq V$. Then $W \leq V$ if and only if the following conditions hold:

1. If $\vec{u}$ and $\vec{v}$ in W, then $\vec{u} \oplus \vec{v} \in W$.
2. If c is any real number and $\vec{u} \in W$, then $c \odot \vec{u} \in W$.

Proof: Let $W \leq V$. Then W is a vector space so that (a) and (b) of the definition of a vector space hold. This means that 1 and 2 of this theorem holds.

For the converse, suppose that 1 and 2 of this theorem holds.

To show: $W \leq V$

Let $\vec{u} \in W$. Then from 2, $(-1) \odot \vec{u} \in W$. Then by 1, $\vec{u} \oplus (-1) \odot \vec{u} \in W$. Since $\vec{u} \oplus (-1) \odot \vec{u} = \vec{0} \in W$. Thus, $\vec{u} \oplus \vec{0} = \vec{u} \in W$. Also, properties (1), (2), (5), (6), (7) and (8) hold in W because they hold in V. Hence, $W \leq V$.

■

EXAMPLE 54. Every vector space V has at least two subspaces: The subspace $\{0\}$ and V itself.

Reason: In any vector space V, $\vec{0} \oplus \vec{0} = \vec{0}$ and $c \odot \vec{0} = \vec{0}$.

The subspace $\{0\}$ is called the zero subspace of V

EXAMPLE 55. Let P_2 be the set consisting of all polynomials of degree less tahn or equal to 2 together with the zero polynomial. Then P_2 is a subset of the vector space P of all polynomials.

It can be shown that P_2 is a subspace of P.

In general, the set P_n of all polynomials of degree less than or equal to n and the zero polynomial is a subspace of P.

Also, P_n is a subspace of P_{n+1}.

EXAMPLE 56. Let V be the set of all polynomials of degree exactly equal to 2. Then V is a subset of the vector space P of all polynomials but V is not a subspace of P.

Example: The sum of polynomials $3x^2 - 2x + 1$ and $-3x^2 + 5x + 1$ is a polynomial of degree 1. It means that the sum of these two polynomials does not belong to V.

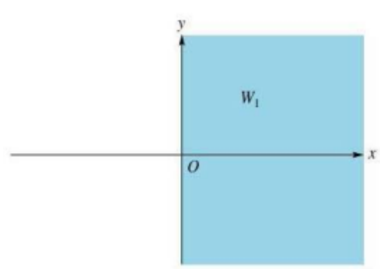
EXAMPLE 57. Determine if the following subsets of R^2 with the usual vector addition and scalar multiplication are subspaces of R^2.

i. $W_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0 \right\}$

ii. $W_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0, y \geq 0 \right\}$

iii. $W_3 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x = 0 \right\}$

Solution. i. W_1 is the right half of the xy-plane.



Take $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in W_1. Then $-2 \odot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$ is not in W_1.

Thus, W_1 is not a subspace of R^2.

ii. W_2 is the first quadrant of the xy-plane. Taking the same vector as in i shows that W_2 is not a subspace of R^2.

iii. W_3 is the y-axis in the xy-plane. It can be shown that W_3 is a subspace of R^2. □

EXAMPLE 58. Let W be the set of all vectors in R^3 of the form $\begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$ where a and b are real numbers. It can be shown that W is a subspace of R^3.

NOTATION:

From now on, we will denote $\vec{u} \oplus \vec{v}$ simply as $u+v$ and $c \odot \vec{u}$ simply as cu if we are in a vector space.

ASSIGNMENT: Show that a nonempty subset W of a vector space V is a subspace of V if and only if $cu + dv \in W$ for every vectors $u, v \in W$ and any scalars c and d.

10.2 Linear Combination

Definition

Let $v_1, v_2, \ldots, v_k$ be vectors in a vector space V. A vector $v \in V$ is called a **linear combination** of the vectors $v_1, v_2, \ldots, v_k$ if

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = \sum_{i=1}^{k} a_i v_i$$

for some real numbers $a_1, a_2, \ldots, a_k$.
Remark: The set S in this definition can be replaced by an infinite set S of vectors in a vector space using corresponding notation for infinite sums.

EXAMPLE 59. In example 58, we have shown that set of all vectors W in R^3 of the form $\begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$ where a and b are real numbers is a subspace of R^3. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Then every vector in W is a linear combination of v_1 and v_2 because $av_1 + bv_2 = \begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$.

EXAMPLE 60. By example 55, P_2 is a vector space of all polynomials of degree 2 or less and the zero polynomial. So each vector in P_2 has the form $ax^2 + bx + c$. This means that every vector in P_2 is a linear combination of vectors x^2, x, and 1.

EXAMPLE 61. Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

The vector $v = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$ is a linear combination of v_1, v_2, v_3 if there are real numbers a_1, a_2, a_3 such that $a_1 v_1 + a_2 v_2 + a_3 v_3 = v$.

Then

$$a_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}.$$

Thus,

$$\begin{aligned} a_1 + a_2 + a_3 &= 2 \\ 2a_1 + a_3 &= 1 \\ a_1 + 2a_2 &= 5. \end{aligned}$$

Solving this linear system gives

$$a_1 = 1, a_2 = 2, a_3 = -1.$$

This means that $v = v_1 + 2v_2 - v_3$ and so v is a linear combination of v_1, v_2 and v_3.

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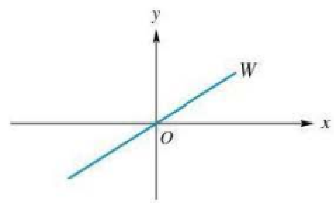
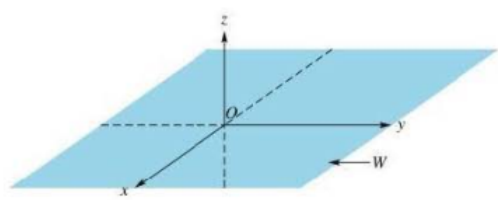
1. The set W consisting of all points in R^2 of the form (x, x) is a straight line. Investigate if W is a subspace of R^2.



2. Let W be the set of all points in R^3 that lie in the xy-plane. Investigate if W is a subspace of R^3.



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Which of the given subsets of R^3 are subspaces? (Explain)

1. The set of all vectors of the form $\begin{bmatrix} a \\ b \\ 1 \end{bmatrix}$.

2. The set of all vectors of the form $\begin{bmatrix} a \\ b \\ a+2b \end{bmatrix}$.

3. The set of all vectors of the form $\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$

4. The set of all vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $a + 2b - c = 0$.

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Which of the following vectors in R^3 are linear combinations of

$$v_1 = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}?$$

1. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2. $\begin{bmatrix} 4 \\ 2 \\ -6 \end{bmatrix}$

3. $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Chapter 11

Span

11.1 Span

Span

If $S = \{v_1, v_2, \ldots, v_k\}$ is a set of vectors in a vector space V, then the set of all vectors in V that are linear combinations of the vectors in S is denoted by

$$\textit{span } S$$

or

$$\textit{span } \{v_1, v_2, \ldots, v_k\}.$$

Remark: This definition also applies to an infinite set S of vectors in a vector space.

EXAMPLE 62. Let

$$S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \right\}$$

Then *span* S is the set in M_{23} consisting of all vectors of the form

$$a \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix},$$

where a, b, c, d are real numbers.

This means that $spanS$ is the subset of M_{23} consisting of all matrices of the form

$$\begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix},$$

where a, b, c, d are real numbers.

EXAMPLE 63. Let $S = \{x^2, x, 1\}$ be a subset of P_2. Then $span S = P_2$.

EXAMPLE 64. Let

$$S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \subseteq R^3.$$

Then *span* S is the set of all vectors in R^3 of the form $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$.

$span\ S$ is a Subspace

THEOREM 17. Let $S = \{v_1, v_2, \dots, v_k\}$ be a set of vectors in a vector space V. Then $span\ S \leq V$.

Proof: Note that $\varnothing \neq spanS \subseteq V$. Let $u, w \in span\ S$. Then

$$u = \sum_{i=1}^{k} a_i v_i$$

and

$$w = \sum_{i=1}^{k} b_i v_i$$

for some $a_i, b_i \in \mathbb{R}$. Thus,

$$u + w = \sum_{i=1}^{k} a_i v_i + \sum_{i=1}^{k} b_i v_i = \sum_{i=1}^{k} (a_i + b_i) v_i.$$

Also,

$$cu = c\left(\sum_{i=1}^{k} a_i v_i \right) = \sum_{i=1}^{k} (ca_i) v_i.$$

Therefore, by subspace criterion, $span\ S \leq V$.

EXAMPLE 65. Let $S = \{x^2, x\} \subseteq P_2$. Then *span* S is the subspace of all polynomials of the form $ax^2 + bx$, where $a, b \in \mathbb{R}$.

EXAMPLE 66. Let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subseteq M_{22}.$$

Then *span* S is the subspace of all 2×2 diagonal matrices.

Spanned by a Set

Let S be a set of vectors in a vector space V. If every vector in V is a linear combination of the vectors in S, then the set S is said to **span** V, or V is spanned by the set S; that is, $span\ S = V$. If $span\ S = V$, then S is called a **spanning set** of V.

EXAMPLE 67. Let $v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$. Then $v_1, v_2 \in R^3$.

Determine whether the vector $v = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}$ belongs to $span\ \{v_1, v_2\}$.

Solution. Let $S = \{v_1, v_2\}$. We can say that $v \in span\ S$ if we can find scalars a and b such that $av_1 + bv_2 = v$. It means that

$$a \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}.$$

This corresponds to a linear system whose augmented matrix is

$$\begin{bmatrix} 2 & 1 & | & 1 \\ 1 & -1 & | & 5 \\ 1 & 3 & | & -7 \end{bmatrix}$$

and whose reduced row echelon form of this system is

$$\begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & -3 \\ 0 & 0 & | & 0 \end{bmatrix}.$$

This means that the linear system is consistent whose $a_1 = 2$ and $a_2 = 3$. Therefore, $v \in span\ S$. □

EXAMPLE 68. Let $v_1 = 2t^2+t+2, v_2 = t^2-2t, v_3 = 5t^2-5t+2, v_4 = -t^2-3t-2$. Then $v_1, v_2, v_3, v_4 \in P_2$. Determine whether the vector $v = t^2 + t + 2$ belongs to $span\ \{v_1, v_2, v_3, v_4\}$.

Solution. In order to say that $v \in span\ S$, we must find scalars a, b, c, d such that $av_1 + bv_2 + cv_3 + dv_4 = v$. This means that

$$a_1(2t^2 + t + 2) + a_2(t^2 - 2t) + a_3(5t^2 - 5t + 2) + a_4(-t^2 - 3t - 2) = t^2 + t + 2.$$

Thus,

$$(2a_1 + a_2 + 5a_3 - a_4)t^2 + (a_1 - 2a_2 - 5a_3 - 3a_4)t + (2a_1 + 2a_3 - 2a_4) = t^2 + t + 2.$$

Equating coefficients of respective powers of t, we get the linear system

$$\begin{aligned} 2a_1 + a_2 + 5a_3 - a_4 &= 1 \\ a_1 - 2a_2 - 5a_3 - 3a_4 &= 1 \\ 2a_1 + 2a_3 - 2a_4 &= 2. \end{aligned}$$

Forming the augmented matrix of this system and transforming it to reduced row echelon form, we obtain

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

This means that the system is inconsistent or has no solution. Hence, v does not belong to *span* S. □

EXAMPLE 69. Let V be the vector space R^3. Let

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Show that $\{v_1, v_2, v_3\}$ span V, i.e, *span* $\{v_1, v_2, v_3\} = R^3$.

Solution. Let $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in R^3$. To show that $\{v_1, v_2, v_3\}$ span V, we determine if there are scalars a_1, a_2, a_3 such that $a_1v_1 + a_2v_2 + a_3v_3 = v$. This will yield to a linear system

$$\begin{aligned} a_1 + a_2 + a_3 &= a \\ 2a_1 + a_3 &= b \\ a_1 + 2a_2 &= c. \end{aligned}$$

A solution to this system is

$$a_1 = \frac{-2a + 2b + c}{3}, a_2 = \frac{a - b + c}{3}, a_3 = \frac{4a - b - 2c}{3}.$$

This means that *span* $S = R^3$. □

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Determine whether the given vector $p(t)$ in P_2 belongs to *span* $\{p_1(t), p_2(t), p_3(t)\}$, where $p_1(t) = t^2 + 2t + 1, p_2(t) = t^2 + 3$, and $p_3(t) = t - 1$.

1. $p(t) = t^2 + t + 2$

2. $p(t) = 2t^2 + 2t + 3$

Chapter 12

Linear Independence

12.1 Linear Independence

In our past lessons, we have shown that the set W of all vectors of the form $\begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$ where a and b are real numbers is a subspace of R^3.

We note also that the following sets are spanning sets for W. (Verify)

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \left\{ \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \right\} \right.\right.$$

$$S_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \left\{ \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right\} \right.\right.\right.$$

$$S_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \right.$$

If we can determine a spanning set for a vector space V that is minimal in the sense that it contains the fewest number of vectors, then we have an efficient way to describe every vector in V.

In this example, the most efficient spanning set is S_3 since it only contains 2 members.

Since the vectors in S_3 span W (means every vector in W is a linear combination of vectors in S_3) and S_3 is a subset of S_1 and S_2, it means that the vector $\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \in S_1$ (which is also in W) is a linear combination of the vectors in S_3. Also, the vectors $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \in S_2$ must be linear combinations of the vectors in S_3.

To see this, we have

$$3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix},$$

$$0 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}.$$

Also for set S_1,

$$3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

For S_2,

$$0 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

OBSERVATION:

If *span* $S = V$ and there is a linear combination of vectors in S with coefficients not all zero that gives the zero vector, then some subset of S is also a spanning set for V.

This motivates our further study.

Linearly Dependent/Independent

The vectors $v_1, v_2, \ldots, v_k$ in a vector space V are said to be **linearly dependent** if there exist constants $a_1, a_2, \ldots, a_k$ which are not all zero such that

$$\sum_{i=1}^{k} a_i v_i = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0.^{**}$$

Otherwise, $v_1, v_2, \ldots, v_k$ are called **linearly independent**. This means that $v_1, v_2, \ldots, v_k$ are linearly independent if

$$\sum_{i=1}^{k} a_i v_i = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0$$

and

$$a_1 = a_2 = \cdots = a_k = 0.$$

We say that $S = \{v_1, v_2, \ldots, v_k\}$ is linearly dependent (independent) if the vectors v_i are linearly dependent (independent.

Remarks:

1. The equation

$$\sum_{i=1}^{k} a_i v_i = a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0$$

in the above definition always holds if all scalars $a_1, a_2, \ldots, a_k$ equal to zero. The important point in this definition is whether it is possible to satisfy (**) with at least one of the scalars different from zero.

2. Regardless or the form of the vectors, equation (**) yields a homogeneous linear system of equations. It is always consistent because $a_i = 0$ for all i is a solution.

EXAMPLE 70. Determine whether the vectors

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

are linearly independent.

Solution.

$$a_1 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From here we get

$$\begin{aligned} 3a_1 + a_2 - a_3 &= 0 \\ 2a_1 + 2a_2 + 2a_3 &= 0 \\ a_1 - a_3 &= 0 \end{aligned}$$

whose augmented matrix is

$$\begin{bmatrix} 3 & 1 & -1 & | & 0 \\ 2 & 2 & 2 & | & 0 \\ 1 & 0 & -1 & | & 0 \end{bmatrix}.$$

Transforming this to its reduced echelon form, we get

$$\begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}.$$

This means that the linear system has a nontrivial solution $\begin{bmatrix} k \\ -2k \\ k \end{bmatrix}$, where $k \neq 0$. Therefore, the vectors v_1, v_2, v_3 are linearly independent. □

PRACTICE:

Are the following vectors linearly dependent or linearly independent?

1. $v_1 = [1\ 0\ 1\ 2], v_2 = [0\ 1\ 1\ 2]$, and $v_3 = [1\ 1\ 1\ 3]$ in R_4
2. $v_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}$ in M_{22}
3. $v_1 = [1\ 0\ 0], v_2 = [0\ 1\ 0], v_3 = [0\ 0\ 1]$ in R_3
4. $v_1 = t^2 + t + 2, v_2 = 2t^2 + t, v_3 = 3t62 + 2t + 2$ in P_2

Linearly Independent Vectors by Checking its Determinant

THEOREM 18. Let $S = \{v_1, v_2, \ldots, v_n\}$ be set of n vectors in R^n (R_n). Let A be the matrix whose columns (rows) are the elements of S. Then S is linearly independent if and only if $det(A) \neq 0$.

EXAMPLE 71. Determine if $S = \{[1\ 2\ 3], [0\ 1\ 2], [3\ 0\ -1]\}$ is a linearly independent set of vectors in R^3.

Solution. Forming the matrix whose rows are the vectors of S, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 0 & -1 \end{bmatrix}$$

whose $det(A) = 2 \neq 0$. Thus, S is linearly independent. □

Linearly Independent/Dependent Subsets

Let S_1 and S_2 be finite subsets of a vector space and let $S_1 \subseteq S_2$. Then the following hold:

1. If S_1 is linearly dependent, then S_2 is also linearly dependent.
2. If S_2 is linearly independent, then S_1 is also linearly independent.

Proof: Let $S_1 = \{v_1, v_2, \ldots, v_k\}$ and $S_2 = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_m\}$. Then $S_1 \subseteq S_2$. We first prove 1. Since S_1 is linearly dependent, there exist a_i's not all zero such that $a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0$. Then

$$a_1v_1 + a_2v_2 + \cdots + a_kv_k + 0v_{k+1} + \cdots + 0v_m = 0.$$

Since not all coefficients of

$$a_1v_1 + a_2v_2 + \cdots + a_kv_k + 0v_{k+1} + \cdots + 0v_m$$

are zero and this equation is a linear combination of vectors of S_2, we say that S_2 is linearly dependent.

Statement 2 is the contrapositive of 1. ■

Remarks:

1. $S = \{0\}$ is linearly dependent. (Example: $3(0) = 0$ and $3 \neq 0$)
2. If S is any set of vectors containing 0, then S must be linearly dependent. (why?)
3. A set of vectors consisting of a single nonzero vector is linearly independent. (why?)
4. If $v_1, v_2, \ldots, v_k$ are vectors in a vector space V and any two of them are equal, then $v_1, v_2, \ldots, v_k$ are linearly dependent. (why?)

Linear Combination of Preceding Vectors

THEOREM 19. The nonzero vectors $v_1, v_2, \ldots, v_n$ in a vector space V are linearly dependent if and only if one of the vectors v_j for $j \geq 2$ is a linear combination of the preceding vectors $v_1, v_2, \ldots, v_{j-1}$.

EXAMPLE 72. Let $V = R^3$ and $v_1 = [1\ 2\ -1], v_2 = [1\ -2\ 1], v_3 = [-3\ 2\ -1]$ and $v_4 = [2\ 0\ 0]$. It can be verified that $v_1 + v_2 + 0v_3 - v_4 = 0$. Thus, v_1, v_2, v_3, v_4 are linearly dependent. By Theorem 19, we can have $v_4 = v_1 + v_2 + 0v_3$.

Remarks:

1. Theorem 19 does not say that each vector v is a linear combination of the preceding vectors.

   In Example 72, we can have $v_1 + 2v_2 + v_3 + 0v_4 = 0$. In this case, we cannot solve for v_4 as linear combination of v_1, v_2 and v_3 since its coefficient is zero.

2. If $S = \{v_1, v_2, \ldots, v_k\}$ is a set of vectors in a vector space V, then S is linearly independent if and only if one of the vectors in S is a linear combination of all other vectors in S. (Problem Set)

   Example: $v_1 = -v_2 - 0v_3 + v_4$ and $v_2 = -\frac{1}{2}v_1 - \frac{1}{2}v_3 - 0v_4$

3. If $\{v_1, v_2, \ldots, v_k\}$ are linearly independent vectors in a vector space, then they must be distinct and nonzero.

THEOREM 20. Suppose that $S = \{v_1, v_2, \ldots, v_k\}$ spans a vector space V and v_j is a linear combination of the preceding vectors in S. Then the set $S_1 = \{v_1, v_2, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\}$ consisting of S with v_j deleted, also spans V.

Proof: Let $v \in V$. Since *span* $S = V$, there exist scalars $a - 1, a_2, \ldots, a_n$ such that

$$v = a_1v_1 + a_2v_2 + \cdots a_{j-1}v_{j-1} + a_jv_j + a_{j+1}v_{j+1} + \cdots + a_nv_n.$$

If $v_j = b_1v_1 + b_2v_2 + \cdots + b_{j-1}v_{j-1}$, then

$$v = a_1v_1 + a_2v_2 + \cdots a_{j-1}v_{j-1} + a_j(b_1v_1 + b_2v_2 + \cdots + b_{j-1}v_{j-1}) + a_{j+1}v_{j+1} + \cdots + a_nv_n$$

and so v can be written as

$$v = c_1v_1 + c_2v_2 + \cdots + c_{j-1}v_{j-1} + c_{j+1}v_{j+1} + \cdots + c_nv_n.$$

This implies that *span* $S_1 = V$. ■

EXAMPLE 73. Let $S = \{v_1, v_2, v_3, v_4\}$ in R^4, where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

and suppose that $W = span\ S$.

Since $v_4 = v_1 + v_2$, we can say that $W = span\ S_1$ where $S_1 = \{v_1, v_2, v_3\}$.

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1. Show that
$$S = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \\ 10 \end{bmatrix} \right\}$$
is a linearly dependent set in R^3.

2. Determine whether
$$S = \{[312], [38-5], [-36-9]\}$$
is a linearly independent set in R_3

3. Which of the given vectors in R_3 are linearly dependent? For those which are, express one vector as a linear combination of the rest.

   (a) $[1\ 1\ 0], [0\ 2\ 3], [1\ 2\ 3], [3\ 6\ 6]$
   (b) $[1\ 1\ 0], [3\ 4\ 2]$
   (c) $[1\ 1\ 0], [0\ 2\ 3], [1\ 2\ 3], [0\ 0\ 0]$

PROBLEM SET

DIRECTION: Write all your answers in a short bondpaper.
DEADLINE: ____________________

1. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of vectors in a vector space V. Prove that S is linearly dependent if and only if one of the vectors in S is a linear combination of all the other vectors in S.

2. Suppose that $S = \{v_1, v_2, v_3\}$ is a linearly independent set of vectors in a vector space V. Prove that $T = \{w_1, w_2, w_3\}$ is also linearly independent, where $w_1 = v_1 + v_2 + v_3, w_2 = v_2 + v_3$ and $w_3 = v_3$.

3. Suppose that $S = \{v_1, v_2, v_3\}$ is a linearly independent set of vectors in a vector space V.

   Is $T = \{w_1, w_2, w_3\}$, where $w_1 = v_1 + v_2, w_2 = v_1 + v_3, w - 3 = v_2 + v_3$, linearly dependent or linearly independent? Justify your answer.

4. Let S_1 and S_2 be finite subsets of a vector space and let $S_1 \subseteq S_2$. If S_2 is linearly dependent, why or why not is S_1 linearly dependent? Give an example.

5. Let S_1 and S_2 be finite subsets of a vector space and let $S_1 \subseteq S_2$. If S_1 is linearly independent, why or why not is S_2 linearly dependent? Give an example.

Chapter 13

Basis and Dimension

In this chapter, we continue our study of the structure of a vector space V by determining a set of vectors in V that completely describes V. We combine in here the topics of span and linear independence.

Basis for a Vector Space

The vectors $v_1, v_2, \ldots, v_k$ in a vector space V are said to form a **basis** for V if

1. *span* $\{v_1, v_2, \ldots, v_k\} = V$ and
2. $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent set.

Remark: If $v_1, v_2, \ldots, v_k$ form a basis for V, then they must be distinct and nonzero. The definition can be applied to an infinite set of vectors in a vector space.

EXAMPLE 74. The vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for R^3. This is called the **natural/standard basis** for R^3.

We can generalize the natural basis for R^n.

In similar way, the vectors $[1\ 0\ 0], [0\ 1\ 0], [0\ 0\ 1]$ form a natural basis for R_3.

EXAMPLE 75. The set $S = \{t^2 + 1, t - 1, 2t + 2\}$ is a basis for the vector space P_2.

Solution. 1. Show: *span* $S = P_2$ Let $v \in P_2$. Then $v = at^2 + bt + c$ where a, b, c are scalars. In order that $v \in$ *span* S, there must exist scalars a_1, a_2, a_3 such that

$$v = at^2 + bt + c = a_1(t^2 + 1) + a_2(t - 1) + a_3(2t + 2).$$

Thus,

$$at^2 + bt + c = a_1 t^2 + (a_2 + 2a_3)t + (a_1 - a_2 + 2a_3).$$

Comparing the coefficients of t^2, t and the constants, we get

$$\begin{aligned} a_1 &= a \\ a_2 + 2a_3 &= b \\ a_1 - a_2 + 2a_3 &= c \end{aligned}$$

Solving this linear system, we have

$$a_1 = a, a_2 = \frac{a+b-c}{2}, a_3 = \frac{c+b-1}{4}.$$

This means that *span* $S = V$.

2. Show: S is linearly independent.

   Write $a_1(t^2+1) + a_2(t-1) + a_3(2t+2) = 0$. Then $a_1 t^2 + (a_2+2a_3)t + (a_1 - a_2 + 2a_3) = 0$. Comparing the coefficients of t^2, t and the constants, we get

$$\begin{aligned} a_1 &= 0 \\ a_2 + 2a_3 &= 0 \\ a_1 - a_2 + 2a_3 &= 0 \end{aligned}$$

   Solving this linear system we have $a_1 = a_2 = a_3 = 0$. This means that S is linearly independent.

Combining 1 and 2, we conclude that S is a basis for P_2. □

EXAMPLE 76. The set of vectors $\{t^n, t^{n-1}, \dots, t, 1\}$ is a natural basis for the vector space P_n.

Remark: Basis for a vector space is not unique based on last two examples.

PRACTICE:

1. The set W of all 2×2 matrices with trace equal to zero is a subspace of M_{22}. Show that $S = \{v_1, v_2, v_3\}$, where

$$v_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

   is a basis for W.

2. Find a basis for the subspace V of P_2 consisting of all vectors of the form $at^2 + bt + c$, where $c = a - b$.

Unique Linear Combination

THEOREM 21. If $S = \{v_1, v_2, \dots, v_n\}$ is a basis for a vector space V, then every vector in V can be written in one and only one way as a linear combination of the vectors in S.

Proof: Let $v \in V$. Since *span* $S = V$, v can be written as a linear combination of vectors in S. Suppose that $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$ and $v = b_1v_1 + b_2v_2 + \cdots + b_nv_n$.

CLAIM: $a_i = b_i$ for all i.

Now,

$$0 = v - v = (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_n - b_n)v_n.$$

Since S is linearly independent, $a_i = b_i$ for all i. ■

THEOREM 22. Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of nonzero vectors in a vector space V and let $W = span\ S$. Then some subset of S is a basis for W.

Proof: *Case 1.* S is linearly independent.

Since S is linearly independent and by assumption *span* $S = W$, S is a basis for W.

Case 2. S is linearly dependent.

Since S is linearly dependent, $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$ where $a_1, a_2, \ldots, a_n$ are not all zero. By Theorem 19, some v_j is a linear combination of the preceding vectors in S. Let $S_1 = S - \{v_j\} \subseteq S$. Then by Theorem 20, *span* $S_1 = W$.

If S_1 is linearly independent, then S_1 is a basis. If S_1 is linearly dependent, then one of the vectors of S_1, say v_j is a linear combination of the preceding vectors.

Let $S_2 = S_1 - \{v_j\}$. Then *span* $S_2 = W$. ■

Steps in Finding a Basis Subset

Let $V = R^m$ or R_m and let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of nonzero vectors in V. The procedure for finding a subset T of S that is a basis for $W = span\ S$ is as follows:

1. Form equation
$$a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0.$$
2. Construct the augmented matrix associated with the homogeneous system of $a_1v_1 + a_2v_2 + \ldots + a_nv_n = 0$ and transform it to reduced row echelon form.
3. The vectors corresponding to the columns containing the leading 1's form a basis T for $W = span\ S$.

EXAMPLE 77. Let $V = R_3$ and $S = \{v_1, v_2, v_3, v_4\}$, where $v_1 = [1\ 0\ 1], v_2 = [0\ 1\ 1], v_3 = [1\ 1\ 2], v_4 = [1\ 2\ 1]$, and $v_5 = [-1\ 1\ -2]$.

It is easy to show that *span* $S = W$. Using above procedure:

Step 1.

$$a_1[1\ 0\ 1] + a_2[0\ 1\ 1] + a_3[1\ 1\ 2] + a_4[1\ 2\ 1] + a_5[-1\ 1\ -2] = [0\ 0\ 0].$$

Step 2. Equating corresponding components, we obtain the homogeneous system

$$\begin{aligned} a_1 + a_3 + a_4 - a_5 &= 0 \\ a_2 + a_3 + 2a_4 + a_5 &= 0 \\ a_1 + a_2 + 2a_3 + a_4 - 2a_5 &= 0. \end{aligned}$$

whose reduced row echelon form for the associated augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right].$$

Step 3. The leading 1's appear in columns 1,2 and 4 so $\{v_1, v_2, v_3\}$ is a basis for R_3.

Some more results:

THEOREM 23.

1. If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V and $T = \{w_1, w_2, \ldots, w_n\}$ is a linearly independent set of vectors in V, then $r \leq n$.
2. If $S = \{v_1, v_2, \ldots, v_n\}$ and $T = \{w_1, w_2, \ldots, w_n\}$ are bases for a vector space V, then $n = m$

Proof: (Assignment) ■

Practice:

I. Which of the following sets of vectors are bases for R^2?

1. $\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$
2. $\left\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\right\}$

II. Find a basis for the subspace W of R^3 spanned by $\left\{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix}\right\}$

PROBLEM SET

DIRECTION: Write all your answers in a short bondpaper.
DEADLINE: ______________________

1. Let $c \neq 0$. If $\{v_1, v_2, \ldots, v_k\}$ is a basis for a vector space V, then If $\{cv_1, cv_2, \ldots, cv_k\}$ is also a basis for a vector space V.

2. Find a basis for the subspace of R^3:

   All vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, where $b = a + c$.

Although a vector space may have many bases, we have just shown that, for a particular vector space V, all bases have the same number of vectors. We can then make the following definition:

Dimension

The dimension of a nonzero vector space V is the number of vectors in a basis for V. We often write *dim* V for the dimension of V. We also define the dimension of the trivial vector space $\{0\}$ to be zero.

EXAMPLE 78. The set $S = \{t^2, t, 1\}$ is a basis for P_2. Thus, *dim* $P_2 = 3$.

Maximal Independent Subset

Let S be a set of vectors in a vector space V. A subset T of S is called a maximal independent subset of S if T is a linearly independent set of vectors that is not properly contained in any other linearly independent subset of S.

EXAMPLE 79. Let $V = R^3$ and consider the set $S = \{v_1, v_2, v_3, v_4\}$, where

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Maximal independent subsets of S are $\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}$ and $\{v_2, v_3, v_4\}$.

Corollary

THEOREM 24. If the vector space V has dimension n, then a maximal independent subset of vectors in V contains n vectors.

Proof: Let $S = \{v_1, v_2, \ldots, v_k\}$ be a maximal independent subset of V. If *span* $S \neq V$, then there exists a vector $v \in V$ that cannot be written as a linear combination of $v_1, v_2, \ldots, v_k$. It follows that $\{v_1, v_2, \ldots, v_k, v\}$ is a linearly independent set of vectors. This is a contradiction to the asusmption that S is a maximal independent subset of V. Thus, *span* $S = V$. This means that S is a basis for V. Therefore, $k = n$. ■

Some more results

THEOREM 25.

1. If a vector space V has dimension n, then a minimal spanning set for V contains n vectors.
2. If a vector space V has dimension n, then any subset of $m > n$ vectors must be linearly dependent.
3. If a vector space V has dimension n, then any subset of $m < n$ vectors cannot span V.

Proof: (Problem Set)

Results:

1. R^3 has dimension 3, R_2 has dimension 2, and R^n and R_n both have dimension n.
2. P_3 has dimension 4 since $\{t^3, t^2, t, 1\}$ is a basis for P_3. In general, P_n has dimension $n + 1$.
3. The subspaces of R^2 are $\{0\}, R^2$ and any line passing through the origin.
4. The subspaces of R^3 are $\{0\}, R^3$ and all lines and planes passing through the origin.

THEOREM 26. If S is a linearly independent set of vectors in a finite-dimensional vector space V, then there is a basis T for V that contains S.

Proof: Let $S = \{v_1, v_2, \ldots, v_m\}$ be a linearly independent set of vectors in the nth dimensional vector space V, where $m < n$. Now let $\{w_1, w_2, \ldots, w_n\}$ be a basis for V. Let $S_1 = \{v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_n\}$. Since *span* S_1, by Theorem 22, it contains a basis T for V. Note that T is obtained by deleting from S_1 every vector that is a linear combination of the preceding vectors. Since S is linearly independent, none of the v_i can be linear combinations of other v_j and thus are not deleted. Hence, T will contain S. ■

EXAMPLE 80. Find a basis for R_4 that contains the vectors $v_1 = [1\ 0\ 1\ 0]$ and $v_2 = [-1\ \ 1\ \ -1\ 0]$.

Solution. Let $\{e_1, e_2, e_3, e_4\}$ be the natural basis for R_4, where

$$e_1 = [1000], e_2 = [0100], e_3 = [0010], e_4 = [0001].$$

Set $S = \{v_1, v_2, e_1, e_2, e_3, e_4\}$. Since $\{e_1, e_2, e_3, e_4\}$ spans R_4, *span* $S = R_4$.

We now find a subset of S that is a basis for R_4.

Form equation

$$a_1v_1 + a_2v_2 + a_3e_3 + a_4e_2 + a_5e_3 + a_6e_4 = [0\ 0\ 0\ 0].$$

This leads to the homogeneous system

$$\begin{aligned} a_1 - a_2 + a_3 &= 0 \\ -a_2 + a_4 &= 0 \\ a_1 - a_2 + a_5 &= 0 \\ a_6 &= 0. \end{aligned}$$

whose augmented matrix transformed to reduced row echelon form is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Since leading 1's appear in columns 1, 2,3, and 6, we say that $\{v_1, v_2, e_1, e_4\}$ is a basis for R_4 containing v_1 and v_2.

□

THEOREM 27. Let V be an n-dimensional vector space. Then

1. If $S = \{v_2, v_2, \ldots, v_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.
2. If $S = \{v_2, v_2, \ldots, v_n\}$ spans V, then S is a basis for V.

Proof: (Problem Set)

EXAMPLE 81. To determine whether a subset of R^n (or R_n) is a basis for R^n (or R_n), we count the number of elements in S. If S has n elements, we use part 1 or 2 of Theorem 27.

If S does not have n elements, it is not a basis for R^n (or R_n). (why?)

For instance, $dim\ R_3 = 3$. Let S contain 4 vectors. Then by Theorem 27, S is not a basis for R_3.

EXAMPLE 82. Note that $dim\ R_4 = 4$. Let S be a set that contains 4 vectors. Then it is possible for S to be a basis for R_4.

If S is linearly independent or spans R_4, then S is a basis.

Recall: If a set S of n vectors in R^n (or R_n) is linearly independent, then S spans R^n (or R_n).

If S spans R^n (or R_n), then S is linearly independent.

This means that the condition in Theorem is also necessary and sufficient for S to span R^n (or R_n).

THEOREM 28. Let S be a finite subset of the vector space V that spans V. A maximal independent subset T of S is a basis for V.

Proof: (Problem Set)

PROBLEM SET

DIRECTION: Write all your answers in a short bondpaper.
DEADLINE: __________________________

1. If a vector space V has dimension n, then a minimal spanning set for V contains n vectors.
2. If a vector space V has dimension n, then any subset of $m > n$ vectors must be linearly dependent.
3. If a vector space V has dimension n, then any subset of $m < n$ vectors cannot span V.
4. Let V be an n-dimensional vector space. If $S = \{v_2, v_2, \ldots, v_n\}$ is a linearly independent set of vectors in V, then S is a basis for V.
5. Let V be an n-dimensional vector space. If $S = \{v_2, v_2, \ldots, v_n\}$ spans V, then S is a basis for V.
6. If W is a subspace of a finite dimensional vector space V, then W is finite dimensional and $dim\ W \leq dim\ V$.
7. Let S be a finite subset of the vector space V that spans V. A maximal independent subset T of S is a basis for V.