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Absolute μS_p-functions and μS_p-Connectedness *

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Abstract

In this paper, the concepts of absolute μS_p-open, absolute μS_p-closed functions, absolute μS_p-continuity, and μS_p-connectedness in generalized topological spaces are introduced and some of their properties are established.

Mathematics Subject Classification: 54A05

Keywords: absolute μS_p-open functions, absolute μS_p-closed functions, absolute μS_p-continuous functions, μS_p-connectedness

1 Introduction

The idea of μS_p-open and μS_p-closed sets in the generalized topological space was introduced in [1]. In order to relate two GT-spaces X and Y, we shall define absolute μS_p-open functions, absolute μS_p-closed functions, and absolute μS_p-continous functions.

Throughout this paper, the space (X, μ) (or simply X) always means a generalized topological space (GT-space) on which no separation axioms are assumed unless explicitly stated. For a subset A of a GT-space X, $\mu S_p c_\mu(A)$, $\mu S_p i_\mu(A)$, and $X \backslash A$ denote the μS_p-closure of A, μS_p-interior of A, and complement of A in X, respectively.

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2 Preliminaries

In [1], Benjamin, P. L and Rara, H. M defined a subset A of a GT-space X to be *μS_p-open* if A is μ-semiopen and for every $x \in A$, there exists a μ-preclosed set F such that $x \in F \subseteq A$. The complement of a μS_p-open set is called a *μS_p-closed* set. The collection of all μS_p-open sets in X forms a strong generalized topology but not always a topology on X and the arbitrary intersection of μS_p-closed sets in X is μS_p-closed. The union of all the μS_p-open sets of a GT-space X contained in A is called the *μS_p-interior of* A, denoted by $\mu S_p i_\mu(A)$. The intersection of all the μS_p-closed sets of X containing A is called the *μS_p-closure* of A, denoted by $\mu S_p c_\mu(A)$.

Definition 2.1 A function $f : (X, \mu_X) \to (Y, \mu_Y)$ is called
(i) *absolute μS_p-open* if the image $f(A)$ is $\mu_Y S_p$-open in Y for each $\mu_X S_p$-open set A in X;
(ii) *absolute μS_p-closed* if the image $f(A)$ is μS_p-closed for each μS_p-closed set A in X;
(iii) *absolute μS_p-continuous* [1] if for every $\mu_Y S_p$-open subset U of Y, $f^{-1}(U)$ is $\mu_X S_p$-open in X;

3 Absolute μS_p-Continuous Functions

In topological spaces, continuous functions send the inverse image of an open set into an open set. The definiton of absolute μS_p-continuous functions seems to be parallel to this. Moreover, its properties behave similarly.

Theorem 3.1 *If $f : (X, \mu_X) \to (Y, \mu_Y)$ and $g : (Y, \mu_Y) \to (Z, \mu_Z)$ are both absolute μS_p-continuous, then $g \circ f : X \to Z$ is absolute μS_p-continuous.*

Proof: Let U be $\mu_Z S_p$-open in Z. Then $g^{-1}(U)$ is $\mu_Y S_p$-open since g is absolute μS_p-continuous. Thus, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is $\mu_X S_p$-open since f is absolute μS_p-continuous. Therefore, $g \circ f$ is absolute μS_p-continous. □

Theorem 3.2 *Let $f : (X, \mu_X) \to (Y, \mu_Y)$ be a function. The following statements are equivalent:*
(i) *f is μS_p-continuous.*
(ii) *For each $x \in X$, and each μ_Y-open set V containing $f(x)$, there exists a $\mu_X S_p$-open set U containing x such that $f(U) \subseteq V$.*
(iii) *$f^{-1}(F)$ is $\mu_X S_p$-closed in X for every μ_Y-closed set F in Y.*
(iv) *$f(\mu_X S_p c_{\mu_X}(A)) \subseteq c_{\mu_Y}(f(A))$ for every $A \subseteq X$.*
(v) *$\mu_X S_p c_{\mu_X}(f^{-1}(B)) \subseteq f^{-1}(c_{\mu_Y}(B))$ for every $B \subseteq Y$.*
(vi) *$f^{-1}(i_{\mu_Y}(B)) \subseteq \mu_X S_p i_{\mu_X}(f^{-1}(B))$ for every $B \subseteq Y$.*

(vii) $i_{\mu_Y}(f(A)) \subseteq f(\mu_X S_p i_{\mu_X}(A))$ *for every subset A of X whenever f is bijective.*

Proof: (i) $\Rightarrow$ (ii): Let $x \in X$ and let V be a μ_Y-open set with $f(x) \in V$. Since f is μS_p-continuous, $f^{-1}(V)$ is $\mu_X S_p$-open in X and $x \in f^{-1}(V)$. Take $U = f^{-1}(V)$ so that $f(U) \subseteq V$ with $x \in U$.
(ii) $\Rightarrow$ (i): Let V be any μ_Y-open set in Y and let $x \in f^{-1}(V)$. Then $f(x) \in V$. By (2), there exists a $\mu_X S_p$-open set U_x such that $x \in U_x$ and $f(U_x) \subseteq V$. Since $\bigcup_{x \in f^{-1}(V)} U_x$ is a $\mu_X S_p$-open set in X, $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ is a $\mu_X S_p$-open set. Therefore, f is μS_p-continuous.
(i) $\Leftrightarrow$ (iii): Let f be a μS_p-continuous function and F be any μ_Y-closed set in Y. Then $Y \backslash F$ is μ_Y-open. Since f is μS_p-continuous, $f^{-1}(Y \backslash F)$ is $\mu_X S_p$-open. Now, $f^{-1}(Y \backslash F) = f^{-1}(Y) \backslash f^{-1}(F) = X \backslash f^{-1}(F)$. Hence, $f^{-1}(F)$ is $\mu_X S_p$-closed in X. Conversely, let F be a μ_Y-open set in Y. Then $Y \backslash F$ is μ_Y-closed. By assumption, $f^{-1}(Y \backslash F)$ is $\mu_X S_p$-closed in X. Since $f^{-1}(Y \backslash F) = X \backslash f^{-1}(F)$, $f^{-1}(F)$ is $\mu_X S_p$-open. Therefore, f is μS_p-continuous.
(iii) $\Rightarrow$ (iv): Let A be any subset of X. Then $f(A) \subseteq c_{\mu_Y}(f(A))$ and $c_{\mu_Y}(f(A))$ is a μ_Y-closed set in Y. By assumption, $f^{-1}(c_{\mu_Y}(f(A)))$ is a $\mu_X S_p$-closed set in X. Hence, $\mu S_p c_{\mu_X}(A) \subseteq f^{-1}(c_{\mu_Y}(f(A)))$. Therefore,

$$f(\mu_X S_p c_{\mu_X}(A)) \subseteq c_{\mu_Y}(f(A)).$$

(iv) $\Rightarrow$ (v): Let $B \subseteq Y$. Then $f^{-1}(B)$ is a subset of X. By (iv),

$$f(\mu_X S_p c_{\mu_X}(f^{-1}(B))) \subseteq c_{\mu_Y} f(f^{-1}(B)) \subseteq c_{\mu_Y}(B).$$

Thus, $\mu_X S_p c_{\mu_X}(f^{-1}(B)) \subseteq f^{-1}(c_{\mu_Y}(B))$.
(v) $\Rightarrow$ (vi): Let $B \subseteq Y$. Since $\mu_X S_p c_{\mu_X}(f^{-1}(Y \backslash B)) = X \backslash \mu_X S_p i_{\mu_X}(f^{-1}(B))$ and $f^{-1}(c_{\mu_Y}(Y \backslash B) = f^{-1}(Y \backslash i_{\mu_Y}(B)) = X \backslash f^{-1}(i_{\mu_Y}(B))$. Applying (v) to $Y \backslash B$, we have $\mu_X S_p c_{\mu_X}(f^{-1}(Y \backslash B)) \subseteq f^{-1}(c_{\mu_Y}(Y \backslash B))$. It follows that

$$f^{-1}(i_{\mu_Y}(B)) \subseteq \mu_X S_p i_{\mu_X}(f^{-1}(B)).$$

(vi) $\Rightarrow$(vii): Let A be any subset of X and f be an injective function. Then by (vi), $f^{-1}(i_{\mu_Y}(f(A))) \subseteq \mu_X S_p i_{\mu_X}(A)$. Therefore, $i_{\mu_Y}(f(A)) \subseteq f(\mu_X S_p i_{\mu_X}(A))$.
(vii) $\Rightarrow$ (i): Let V be a μ_Y-open subset of Y and f be a surjective function. Then by (vii), $i_{\mu_Y}(f(f^{-1}(V))) \subseteq f(\mu_X S_p i_{\mu_X}(f^{-1}(V)))$. Thus,

$$i_{\mu_Y}(V) \subseteq f(\mu_X S_p i_{\mu_X}(f^{-1}(V))).$$

Since V is μ_Y-open, $V \subseteq f(\mu_X S_p i_{\mu_X}(f^{-1}(V)))$ so that

$$f^{-1}(V) \subseteq \mu_X S_p i_{\mu_X}(f^{-1}(V)).$$

Hence, $\mu_X S_p i_{\mu_X}(f^{-1}(V)) = f^{-1}(V)$ which is $\mu_X S_p$-open. Therefore, f is μS_p-continuous. The proof is complete. □

Remark 3.3 Let $(X, \mathscr{P}(X))$ be a GT-space. Then A is μS_p-open for every $A \subseteq X$. In particular, in the space $\mathit{2} = (\{0,1\}, \mathscr{P}(\{0,1\}))$, every subset of $\{0,1\}$ is μS_p-open.

Theorem 3.4 *Let X be a GT-space and let $\chi_A : X \to \mathit{2}$ be the characteristic function of a subset A of X. Then χ_A is absolute μS_p-continuous if and only if A is both μS_p-open and μS_p-closed.*

Proof: Suppose that χ_A is absolute μS_p-continuous. Let $O_1 = \{1\}$ and $O_2 = \{0\}$. Then O_1 and O_2 are μS_p-open in $\{0,1\}$. Since χ_A is absolute μS_p-continuous, $\chi_A^{-1}(O_1) = A$ and $\chi_A^{-1}(O_2) = X \backslash A$ are μS_p-open sets in X. Thus, A is both μS_p-open and μS_p-closed.

Conversely, let A be both μS_p-open and μS_p-closed in X. Let O be a μS_p-open set in $\{0,1\}$. Then

$$\chi_A^{-1}(O) = \begin{cases} \varnothing & \text{if } O = \varnothing, \\ X & \text{if } O = \{0,1\}, \\ A & \text{if } O = \{1\}, \\ X \backslash A & \text{if } O = \{0\}. \end{cases}$$

It means that $\chi_A^{-1}(O)$ is μS_p-open. Therefore, χ_A is absolute μS_p-continuous. This completes the proof. □

Theorem 3.5 *Let $f : (X, \mu_X) \to (Y, \mu_Y)$ and $g : (Y, \mu_Y) \to (Z, \mu_Z)$ be mappings such that the composition $g \circ f : X \to Z$ is absolute μS_p-closed. If f is absolute μS_p-continuous and surjective, then g is absolute μS_p-closed.*

Proof: Let f be absolute μS_p-continuous and surjective and let A be a $\mu_Y S_p$-closed subset of Y. Since f is absolute μS_p-continuous, $f^{-1}(A)$ is $\mu_X S_p$-closed in X. Since $g \circ f$ is μS_p-closed, $(g \circ f)(f^{-1}(A))$ is $\mu_Z S_p$-closed in Z. Since f is surjective, $(g \circ f)(f^{-1}(A)) = g(f(f^{-1}(A))) = g(A)$ is also μS_p-closed. Therefore, $g(A)$ is an $\mu_Z S_p$-closed set in Z and g is an absolute μS_p-closed function. □

4 Absolute μS_p-open and Absolute μS_p-closed Functions

This section includes some properties of absolute μS_p-open and absolute μS_p-closed functions.

Theorem 4.1 *Let $f : (X, \mu_X) \to (Y, \mu_Y)$ be a bijective function. Then the following statements are equivalent:*

1. *f is absolute μS_p-open.*

2. *f is absolute μS_p-closed.*

3. *$f(\mu_X S_p i_{\mu_X}(A)) \subseteq \mu_Y S_p i_{\mu_Y}(f(A))$ for every $A \subseteq X$.*

4. *For each subset W of Y and each $\mu_X S_p$-open set U containing $f^{-1}(W)$, there exists a $\mu_Y S_p$-open set V of Y such that $W \subseteq V$ and $f^{-1}(V) \subseteq U$.*

5. *For every subset S of Y and for every $\mu_X S_p$-closed set F of X containing $f^{-1}(S)$, there exists a $\mu_Y S_p$-closed set K of Y containing S such that $f^{-1}(K) \subseteq F$.*

6. *$f^{-1}(\mu_Y S_p c_{\mu_Y}(B)) \subseteq \mu_X S_p c_{\mu_X}(f^{-1}(B))$ for every subset B of Y.*

7. *$\mu_Y S_p c_{\mu_Y}(f(A)) \subseteq f(\mu_X S_p c_{\mu_X}(A))$ for every subset A of X.*

Proof:
(1) $\Leftrightarrow$ (2): Let f be μS_p-open and D be $\mu_X S_p$-closed in X. Then $X\backslash D$ is $\mu_X S_p$-open and $f(X\backslash D)$ is $\mu_Y S_p$-open. Since f is bijective, $Y\backslash f(D) = f(X\backslash D)$ is $\mu_Y S_p$-open. Thus, $f(D)$ is $\mu_Y S_p$-closed.

Conversely, let f be μS_p-closed and suppose that O is a $\mu_X S_p$-open set in X. Then $X\backslash O$ is $\mu_X S_p$-closed and $f(X\backslash O) = Y\backslash f(O)$ is $\mu_Y S_p$-closed. Therefore, $f(O)$ is $\mu_Y S_p$-open.
(1) $\Leftrightarrow$ (3): Let $A \subseteq X$ and suppose that f is absolute μS_p-open. Since $\mu_X S_p i_{\mu_X}(A)$ is $\mu_X S_p$-open and f is absolute μS_p-open, $f(\mu_X S_p i_{\mu_X}(A))$ is $\mu_Y S_p$-open. Also, $\mu_X S_p i_{\mu_X}(A) \subseteq A$ implies that $f(\mu_X S_p i_{\mu_X}(A)) \subseteq f(A)$. Thus, $f(\mu_X S_p i_{\mu_X}(A)) \subseteq \mu_Y S_p i_{\mu_Y}(f(A))$ by definition of $\mu_Y S_p i_{\mu_Y}(f(A))$.

Conversely, let O be a $\mu_X S_p$-open set in X. Then $\mu_X S_p i_{\mu_X}(O) = O$ and $f(\mu_X S_p i_{\mu_X}(O)) = f(O) \subseteq \mu_Y S_p i_{\mu_Y}(f(O)) \subseteq f(O)$. Hence, $\mu_Y S_p i_{\mu_Y}(f(O)) = f(O)$. Since $\mu_Y S_p i_{\mu_Y}(f(O))$ is $\mu_Y S_p$-open, $f(O)$ is $\mu_Y S_p$-open. Therefore, f is an absolute μS_p-open function.
(2) $\Leftrightarrow$ (7): Let $A \subseteq X$ and suppose that f is absolute μS_p-closed. Since $A \subseteq c_{\mu_X}(A)$, $f(A) \subseteq f(\mu_X S_p c_{\mu_X}(A))$. Moreover, since $\mu_X S_p c_{\mu_X}(A)$ is $\mu_X S_p$-closed in X, $f(\mu_X S_p c_{\mu_X}(A))$ is $\mu_Y S_p$-closed. Therefore, $\mu_Y S_p c_{\mu_Y}(f(A)) \subseteq f(\mu_X S_p c_{\mu_X}(A))$.

Conversely, let O be $\mu_X S_p$-closed. Then $\mu_X S_p c_{\mu_X}(O) = O$ and $f(\mu_X S_p c_{\mu_X}(O)) = f(O)$. Since $f(O) \subseteq \mu_Y S_p c_{\mu_Y}(f(O)) \subseteq f(\mu_X S_p c_{\mu_X}(O)) = f(O)$, $\mu_Y S_p c_{\mu_Y}(f(O)) = f(O)$. Since $\mu_Y S_p c_{\mu_Y}(f(O))$ is $\mu_Y S_p$-closed, $f(O)$ is $\mu_Y S_p$-closed. Therefore, f is an absolute μS_p-closed function.
(1) $\Leftrightarrow$ (5): Suppose that f is absolute μS_p-open. Let $S \subseteq Y$ and F be a $\mu_X S_p$-closed subset of X such that $f^{-1}(S) \subseteq F$. Now, $X\backslash F$ is a $\mu_X S_p$-open set in X. Since f is absolute μS_p-open, $f(X\backslash F)$ is $\mu_Y S_p$-open in Y. Then $K = Y\backslash f(X\backslash F)$ is a $\mu_Y S_p$-closed set in Y. Since $f^{-1}(S) \subseteq F$, $X\backslash F \subseteq$

$X\backslash f^{-1}(S) = f^{-1}(Y\backslash S)$. Thus, $f(X\backslash F) \subseteq f(f^{-1}(Y\backslash S)) \subseteq Y\backslash S$. Hence $Y\backslash(Y\backslash S) \subseteq Y\backslash f(X\backslash F)$ implying that $S \subseteq K$ and

$$f^{-1}(K) = X\backslash f^{-1}(f(X\backslash F)) \subseteq X\backslash(X\backslash F) = F.$$

For the converse, let U be a $\mu_X S_p$-open set in X. Since $X\backslash U$ is $\mu_X S_p$-closed and $f^{-1}(Y\backslash f(U)) = X\backslash(f^{-1}(f(U))) \subseteq X\backslash U$, by assumption, there exists a $\mu_Y S_p$-closed subset K of Y such that $Y\backslash f(U) \subseteq K$ and $f^{-1}(K) \subseteq X\backslash U$ so that $U \subseteq X\backslash f^{-1}(K)$. Hence, $Y\backslash K \subseteq f(U) \subseteq f(X\backslash f^{-1}(K)) \subseteq Y\backslash K$. This implies that $f(U) = Y\backslash K$. Since $Y\backslash K$ is $\mu_Y S_p$-open, $f(U)$ is $\mu_Y S_p$-open in Y. Therefore, f is absolite μS_p-open.
(2) $\Leftrightarrow$ (4): Similar to (1) $\Leftrightarrow$ (5).
(1) $\Leftrightarrow$ (6): Suppose that $f : X \to Y$ is an absolute μS_p-open function and let B be any subset of Y. Since $f^{-1}(B) \subseteq c_{\mu_X}(f^{-1}(B))$ and $\mu_X S_p c_{\mu_X}(f^{-1}(B))$ is $\mu_X S_p$-closed in X, by (1) $\Leftrightarrow$ (5), there exists a $\mu_Y S_p$-closed set K of Y such that $B \subseteq K$ and $f^{-1}(K) \subseteq c_{\mu_X}(f^{-1}(B))$. Hence, $\mu_Y S_p c_{\mu_Y}(B) \subseteq K$. Therefore, $f^{-1}(\mu_Y S_p c_{\mu_Y}(B)) \subseteq f^{-1}(K) \subseteq \mu_X S_p c_{\mu_X}(f^{-1}(B))$.

Conversely, let O be a μ_X-open set in X. Then $X\backslash O$ is μ_X-closed and $f^{-1}(\mu_Y S_p c_{\mu_Y}(f(X\backslash O))) \subseteq X\backslash O$. Also, $X\backslash O \subseteq f^{-1}(\mu_Y S_p c_{\mu_Y}(f(X\backslash O)))$ and $\mu_Y S_p c_{\mu_Y}(f(X\backslash O)) = Y\backslash f(O)$. Since $\mu_Y S_p c_{\mu_Y}(f(X\backslash O))$ is $\mu_Y S_p$-closed, $f(O)$ is $\mu_Y S_p$-open. Therefore, f is an absolute μS_p-open function. $\square$

Theorem 4.2 *If $f : X \to Y$ and $g : Y \to Z$ are both absolute μS_p-open functions, then the composition $g \circ f : X \to Z$ is absolute μS_p-open.*

Proof: Let F be any μS_p-open set in X. Since f is absolute μS_p-open, $f(F)$ is μS_p-open in Y. Because g is absolute μS_p-open, $g(f(F))$ is μS_p-open in Z. Thus, $(g \circ f)(F) = g(f(F))$ is μS_p-open and hence $g \circ f$ is μS_p-open. $\square$

Theorem 4.3 *For a bijection map $f : X \to Y$, the following are equivalent:*

(a) *$f^{-1} : Y \to X$ is absolute μS_p-continuous.*

(b) *f is absolute μS_p-open.*

(c) *f is absolute μS_p-closed.*

Proof: (a)$\Rightarrow$(b): Let U be a μS_p-open set of X. By hypothesis, $(f^{-1})^{-1}(U) = f(U)$ is μS_p-open in Y so that f is μS_p-open.
(b)$\Rightarrow$(c): Let F be a μS_p-closed set of X. Then $X\backslash F$ is μS_p-open in X.
By assumption, $f(X\backslash F)$ is μS_p-open in Y. Since f is bijective, $X\backslash f(F) = f(X\backslash F)$ is μS_p-open in Y. Hence, $f(F)$ is μS_p-closed in Y. Therefore, f is μS_p-closed.
(c)$\Rightarrow$(a): Let F be a μS_p-closed set of X. By (c), $f(F)$ is μS_p-closed in Y. But $f(F) = (f^{-1})^{-1}(F)$. Thus, f^{-1} is μS_p-continuous. $\square$

5 μS_p-connectedness

Definition 5.1 A GT-space (X, μ) is *μS_p-connected* if it is not the union of two nonempty disjoint μS_p-open sets. Otherwise, the space (X, μ) is μS_p-disconnected.

Remark 5.2 *A space (X, μ) is μS_p-disconnected if there exist a disjoint nonempty μS_p-open sets A and B such that $X = A \cup B$. The set $A \cup B$ is called the μS_p-decomposition of X.*

Theorem 5.3 *Let (X, μ) be a GT-space. Then the following statements are equivalent:*

(a) *X is μS_p-connected.*

(b) *The only subsets of X both μS_p-open and μS_p-closed are $\varnothing$ and X.*

(c) *No absolute μS_p-continuous function $f : X \to 2$ is surjective, where 2 is the space $\{0, 1\}$ with the discrete topology.*

Proof: (a)⇒(b) Let G be both μS_p-open and μS_p-closed set in X and suppose that $G \neq \varnothing, X$. Then $G \cup X \backslash G$ is an μS_p-decomposition of X. It follows that X is not μS_p-connected. Thus, the only subsets of X both μS_p-open and μS_p-closed are $\varnothing$ and X.
(b)⇒(c) Suppose that $f : X \to 2$ is μS_p-continuous and surjective. Then $f^{-1}(\{0\}) \neq \varnothing, X$. Since $\{0\}$ is both μS_p-open and μS_p-closed in *2*, $f^{-1}(\{0\})$ is both μS_p-open and μS_p-closed. This is a contradiction to our hypothesis. Thus, no μS_p-continuous function $f : X \to 2$ is surjective.
(c)⇒(a) Suppose that X is μS_p-disconnected. Then $X = A \cup B$, where A and B are disjoint nonempty μS_p-open sets. It follows that A and B are also μS_p-closed sets in X. Now, consider the characteristic function χ_A. By Theorem 3.4, χ_A is absolute μS_p-continuous and surjective. This contradicts our assumption. Therefore, A is μS_p-connected. □

Theorem 5.4 *The absolute μS_p-continuous image of an μS_p-connected space is μS_p-connected.*

Proof: Let X be a μS_p-connected space and let $f : X \to f(X)$ be an absolute μS_p-continuous function. Suppose that $f(X)$ is μS_p-disconnected. Then there exists an absolute μS_p-continuous surjection $g : f(X) \to 2$ by Theorem 5.3. By Theorem 3.1, the composition of two absolute μS_p-continuous functions is absolute μS_p-continuous. Thus, $g \circ f : X \to 2$ is an absolute μS_p-continuous surjection which is a contradiction to Theorem 5.3. Therefore, $f(X)$ is μS_p-connected. □

Theorem 5.5 *The union of any family of μS_p-connected GT-spaces having at least one point in common is also μS_p-connected.*

Proof: Let $\{X_\alpha : \alpha \in I\}$ be a collection of μS_p-connected sets in X, and let $X = \cup_\alpha X_\alpha$, where X_α are μS_p-connected for each α. Suppose that $x_o \in \cap_\alpha X_\alpha$ and $f : X \to 2$ be an absolute μS_p-continuous function. Since each X_α is μS_p-connected, $f|_{A_\alpha}$ is not surjective. Moreover, since $x_o \in \cap_\alpha X_\alpha$, $f(x) = f(x_o)$ for all $x \in X_\alpha$ for each α. Therefore, f cannot be surjective. By Theorem 5.3, $X = \cup_\alpha X_\alpha$ is μS_p-connected. □

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Rw-Connectedness and rw-Sets in the Product Space [1]

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Abstract

In this paper, the concept of rw-connectedness and rw-sets in the product space is studied. Specifically, this paper characterized rw-connectedness in terms of rw-open and rw-closed sets and rw-continuous functions. This also established some results involving regular open, regular semiopen, rw-interior, and rw-closed sets in the product of subsets of a topological space.

Mathematics Subject Classification: 54A05

Keywords: rw-open functions, rw-closed functions, rw-connectedness

1 Introduction

In 1937, Stone [6] introduced and investigated the regular open sets. These sets are contained in the family of open sets since a set is regular open if it is equal to the interior of its closure. In 1978, Cameron [2] also introduced and investigated the concept of a regular semiopen set. A set A is regular semiopen if there is a regular open set U such that $U \subseteq A \subseteq \overline{U}$. In 2007, a new class of sets called regular w-closed sets (rw-closed sets) was introduced by Benchalli

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and Wali [1]. A set B is rw-closed if $\overline{B} \subseteq U$ whenever $B \subseteq U$ for any regular semiopen set U. They proved that this new class of sets is properly placed in between the class of w-closed sets [5] and the class of regular generalized closed sets [4].

In this paper, the concepts of rw-connectedness and rw-open sets in the product space are further investigated.

Throughout this paper, space (X, T) (or simply X) always means a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X, $\overline{A}$, $int(A)$, and $\mathrm{C}(A)$ denote the closure of A, interior of A, and complement of A in X, respectively.

2 Preliminaries

Definition 2.1 [1] A function $f : X \to Y$ is called
(i) *rw-open* if the image $f(A)$ is rw-open in Y for each open set A in X.
(ii) *rw-closed* if the image $f(A)$ is rw-closed for each closed set A in X.
(iii) *rw-continuous* if for every open subset U of Y, $f^{-1}(U)$ is rw-open in X.
(iv) *regular strongly continuous* (briefly rs-continuous) if the inverse image of every rw-open set in Y is open in X, that is, $f^{-1}(A)$ is open in X for all rw-open sets A in Y.

3 rw-connectedness

Definition 3.1 A space (X, T) is *rw-connected* if it is not the union of two nonempty disjoint rw-open sets. Otherwise, a space (X, T) is rw-disconnected. A subset A of a topological space is rw-connected if it is rw-connected as a subspace of X.

Remark 3.2 *A space (X, T) is rw-disconnected if there exist a disjoint nonempty rw-open sets A and B such that $X = A \cup B$. The set $A \cup B$ is called the rw-decomposition of X.*

Theorem 3.3 *Let X be any space and let $\chi_A : X \to 2$ be the characteristic function of a subset A of X. Then χ_A is rw-continuous if and only if A is both rw-open and rw-closed.*

Proof: Suppose that χ_A is rw-continuous. Let $O_1 = \{1\}$ and $O_2 = \{0\}$. Then O_1 and O_2 are open in $\{0, 1\}$. Since χ_A is rw-continuous, $\chi_A^{-1}(O_1) = A$ and $\chi_A^{-1}(O_2) = \mathrm{C}(A)$ are rw-open sets in X. Thus, A is both rw-open and rw-closed.

Conversely, let A be both rw-open and rw-closed in X. Let O be an open set in $\{0, 1\}$. Then

$$\chi_A^{-1}(O) = \begin{cases} \varnothing & \text{if } O = \varnothing \\ X & \text{if } O = \{0, 1\} \\ A & \text{if } O = \{1\} \\ \mathrm{C}(A) & \text{if } O = \{0\}. \end{cases}$$

It means that $\chi_A^{-1}(O)$ is rw-open. Therefore, χ_A is rw-continuous. □

Theorem 3.4 *Let* (X, T) *be a topological space. Then the following statements are equivalent:*

(a) X *is* rw*-connected.*

(b) *The only subsets of* X *both* rw*-open and* rw*-closed are* $\varnothing$ *and* X*.*

(c) *No* rw*-continuous function* $f : X \to 2$ *is surjective, where 2 is the space* $\{0, 1\}$ *with the discrete topology.*

Proof: (a)⇒(b) Let G be both rw-open and rw-closed set in X and suppose that $G \neq \varnothing, X$. Then $G \cup \mathrm{C}(G)$ is an rw-decomposition of X. It follows that X is not rw-connected. Thus, the only subsets of X both rw-open and rw-closed are $\varnothing$ and X.
(b)⇒(c) Suppose that $f : X \to 2$ is rw-continuous and surjective. Then $f^{-1}(\{0\}) \neq \varnothing, X$. Since $\{0\}$ is both open and closed in *2*, $f^{-1}(\{0\})$ is both rw-open and rw-closed. This is a contradiction to our hypothesis. Thus, no rw-continuous function $f : X \to 2$ is surjective.
(c)⇒(a) Suppose that X is rw-disconnected. Then $X = A \cup B$, where A and B are disjoint nonempty rw-open sets. It follows that A and B are also rw-closed sets in X. Now, consider the characteristic function χ_A. By Theorem 3.3, χ_A is rw-continuous and surjective. This contradicts our assumption. Therefore, A is rw-connected. □

Theorem 3.5 *Every* rw*-connected space is connected.*

Proof: Suppose that a space X is rw-connected and X is not connected. Then there exist two nonempty disjoint open sets O_1 and O_2 such that $X = O_1 \cup O_2$. Thus X is also the union of two nonempty disjoint rw-open sets. Thus, X is not rw-connected which is a contradiction. Therefore, X is connected. □

Remark 3.6 *The converse of Theorem* 3.5 *is not true.*

To see this, consider the space (X, T) where $X = \{a, b, c\}$ and $\mathrm{T} = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. Then the possible decomposition of X is $\{a, b\} \cup \{c\}$ but $\{c\}$ is not open. Thus, X is connected. The rw-open sets in X are $X, \varnothing, \{a\}, \{b\}, \{c\}$, and $\{a, b\}$. Now, $X = \{a, b\} \cup \{c\}$ implying that X is rw-disconnected.

Theorem 3.7 *The rw-continuous image of an rw-connected set is connected.*

Proof: Let X be an rw-connected set and let $f : X \to f(X)$ be an rw-continuous function. Supopose that $f(X)$ is disconnected. Then by there exists a continuous surjection $g : f(X) \to \mathit{2}$. Hence, $g \circ f : X \to \mathit{2}$ is an rw-continuous surjection which is a contradiction to Theorem 3.4. Therefore, $f(X)$ is connected. □

4 rw-sets in the Product Space

Throughout this section, let $\{Y_\alpha |\ \alpha \in \mathrm{A}\}$ be family of topological spaces, $\prod\{Y_\alpha |\ \alpha \in \mathrm{A}\}$ be the cartesian product space, A_i and B_i are subsets of Y_i.

Theorem 4.1 *If A and B are subsets of X with $A \subseteq B$, then rw-$(\overline{A}) \subseteq rw$-$(\overline{B})$.*

Lemma 4.2 *$\prod_{i=1}^{n} B_i$ is regular open if and only if B_i is regular open for every $i = 1, 2, ..., n$.*

Proof: Let $\prod_{i=1}^{n} B_i$ be a regular open set. Then

$$int\left(\overline{\prod_{i=1}^{n} B_i}\right) = int\left(\prod_{i=1}^{n} \overline{B_i}\right) = \prod_{i=1}^{n} int(\overline{B_i}) = \prod_{i=1}^{n} B_i.$$

Therefore, $int(\overline{B_i}) = B_i$. Hence, B_i is regular open.
The converse is proved similarly. □

Lemma 4.3 *If A_i is regular semiopen for every $i = 1, 2, ..., n$, then $\prod_{i=1}^{n} A_i$ is regular semiopen.*

Proof: Let A_i be regular semiopen for every $i = 1, 2, ..., n$. Then there exists a regular open U_i such that $U_i \subseteq A_i \subseteq \overline{U_i}$. By Theorem 4.2, $\prod_{i=1}^{n} U_i$ is regular open and
$\prod_{i=1}^{n} U_i \subseteq \prod_{i=1}^{n} A_i \subseteq \prod_{i=1}^{n} \overline{U_i} = \overline{\prod_{i=1}^{n} U_i}$. Therefore $\prod_{i=1}^{n} A_i$ is regular semiopen. □

Remark 4.4 *If A is regular open (regular semiopen) in $\prod_{i=1}^{n} Y_i$, then A is not necessarily a cartesian product of regular open (regular semiopen) sets in Y_i.*

Lemma 4.5 *If $\prod_{i=1}^{n} F_i$ is rw-closed in $\prod_{i=1}^{n} X_i$, then F_i is rw-closed in X_i for every $i = 1, 2, ..., n$.*

Proof: Suppose that $\prod_{i=1}^{n} F_i$ is rw-closed in $\prod_{i=1}^{n} X_i$ and let $F_i \subseteq U_i$ where U_i is regular semiopen. Then $\prod_{i=1}^{n} F_i \subseteq \prod_{i=1}^{n} U_i$. Since $\prod_{i=1}^{n} F_i$ is rw-closed and $\prod_{i=1}^{n} U_i$ is regular semiopen by Lemma 4.3, $\overline{\prod_{i=1}^{n} F_i} \subseteq \prod_{i=1}^{n} U_i$. But $\prod_{i=1}^{n} \overline{F_i} = \overline{\prod_{i=1}^{n} F_i} \subseteq \prod_{i=1}^{n} U_i$ implies that $\overline{F_i} \subseteq U_i$ for every $i = 1, 2, ..., n$. Therefore, F_i is rw-closed for every $i = 1, 2, ..., n$. □

Lemma 4.6 $rw\text{-}int(A) = \mathrm{C}(rw\text{-}(\overline{\mathrm{C}(A)})$.

Proof:

$$\begin{aligned} x \in rw\text{-}int(A) &\Leftrightarrow x \in O \text{ for some } rw\text{-open set } O \text{ with } O \subseteq A \\ &\Leftrightarrow x \notin \mathrm{C}(O) \text{ for some } rw\text{-closed set } \mathrm{C}(O) \\ &\quad\ \ \text{with } \mathrm{C}(A) \subseteq \mathrm{C}(O) \\ &\Leftrightarrow x \notin rw\text{-}(\overline{\mathrm{C}(A)}) \\ &\Leftrightarrow x \in \mathrm{C}(rw\text{-}(\overline{\mathrm{C}(A)})) \end{aligned}$$

This completes the proof. □

Lemma 4.7 $rw\text{-}int(A) = \mathrm{C}(rw\text{-}(\overline{\mathrm{C}(A)})$.

Proof:

$$\begin{aligned} x \in rw\text{-}int(A) &\Leftrightarrow x \in O \text{ for some } rw\text{-open set } O \text{ with } O \subseteq A \\ &\Leftrightarrow x \notin \mathrm{C}(O) \text{ for some } rw\text{-closed set } \mathrm{C}(O) \\ &\quad\;\; \text{with } \mathrm{C}(A) \subseteq \mathrm{C}(O) \\ &\Leftrightarrow x \notin rw\text{-}(\overline{\mathrm{C}(A)}) \\ &\Leftrightarrow x \in \mathrm{C}(rw\text{-}(\overline{\mathrm{C}(A)})) \end{aligned}$$

This completes the proof. □

Theorem 4.8 $rw\text{-}int\left(\prod_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} rw\text{-}int(A_i)$.

Proof: By Lemma 4.7, and Theorem 4.1,

$$\begin{aligned} rw\text{-}int\left(\prod_{i=1}^{n} A_i\right) &= \mathrm{C}\left(rw\text{-}\left(\overline{\mathrm{C}(\prod_{i=1}^{n} A_i)}\right)\right) \\ &= \mathrm{C}\left(rw\text{-}\left(\overline{\bigcup_{i=1}^{n} \langle \mathrm{C}(A_i)\rangle}\right)\right) \\ &= \mathrm{C}\left(\bigcup_{i=1}^{n} rw\text{-}(\overline{\langle \mathrm{C}(A_i)\rangle})\right) \\ &= \bigcap_{i=1}^{n} \mathrm{C}(rw\text{-}(\overline{\langle \mathrm{C}(A_i)\rangle})) \\ &= \bigcap_{i=1}^{n} \left\langle \mathrm{C}(rw\text{-}(\overline{\mathrm{C}(A_i)}))\right\rangle \\ &= \bigcap_{i=1}^{n} \langle rw\text{-}int(A_i)\rangle \\ &= \prod_{i=1}^{n} rw\text{-}int(A_i). \end{aligned}$$

□

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Duality of codes over a certain ring of order 2^m

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Abstract: We introduce a non-unital and non-commutative ring $S_m(\mathbb{F}_1)$, called ring of ordered sum over $\mathbb{F}_2$, the binary field. We discuss linear codes over this ring, also known as S_m-codes, and their algebraic structure, particularly, their residue and torsion codes. We explore the generalized notion of duality of S_m-codes.

Keywords Self-orthogonal codes, Self-dual codes, Quasi self-dual codes, Type IV codes, Non-unital ring

1 INTRODUCTION

Self-dual codes and self-orthogonal codes, and consequently, Type IV codes, which are self-dual codes where all the codewords have even weight, have been studied extensively for their vast applications. Many examples of these types of codes have good parameters. Classically, these codes are defined over finite fields. Recently, there have been great interest in codes over finite rings. However, these rings are often commutative, and most of the time, unital [?, ?, ?]. If the ring is noncommutative and without the unity, the usual notion of duality as in finite fields and other commutative rings [?, ?] have to be reconsidered. In particular, left and right duals need to be defined, as in quasi-self dual (QSD) codes.

In this paper, we introduce the ring $S_m(\mathbb{F}_2)$, called the ring of ordered sum over the binary field $\mathbb{F}_2$, defined as

$$S_m(\mathbb{F}_2) = \{(a_1, a_2 \dots, a_m) | a_1, a_2 \dots, a_m \in \mathbb{F}_2\}$$

together with the following binary operations, addition and multiplication respectively,

$$(a_1, \dots, a_m) + (b_1, \dots, b_m) = (a_1 + b_1, \dots, a_m + b_m),$$

$$(a_1, \dots, a_m) \cdot (b_1, \dots, b_m) = \left(a_1 \sum_{i=1}^{m} b_i, \dots, a_m \sum_{i=1}^{m} b_i \right).$$

We call linear codes over this ring simply as S_m-codes. We will redefine the notion of duality of S_m-codes. Moreover, for an S_m-code C, we associate binary codes called residue and ith torsion, for $i = 1, 2, \dots, m-1$. We then study the structure of QSD codes of length n, defined as self-orthogonal codes of size $2^{\frac{mn}{2}}$ and Type IV codes, defined as QSD codes with all codewords of even Hamming weight, in terms of their residue and torsion codes. The conditions for the existence of these codes will be given.

2 PRELIMINARIES

2.1 THE RING $S_m(\mathbb{F}_2)$

In this section, we give some basic properties of the ring $S_m(\mathbb{F}_2)$.

Theorem 1. *Let*

$$O_m(\mathbb{F}_2) = \{(a_1, a_2, \dots, a_m) \in S_m(\mathbb{F}_2) \mid \sum_{i=1}^{m} a_i = 0\}.$$

Then $O_m(\mathbb{F}_2)$ is a commutative ideal of $S_m(\mathbb{F}_2)$ and $S_m(\mathbb{F}_2)/O_m(\mathbb{F}_2) \cong \mathbb{F}_2$.

The ideals of $S_m(\mathbb{F}_2)$ can be characterized as follows.

Proposition 1. *For positive integer m, $S_m(\mathbb{F}_2)$ has ideal J_i of size 2^{m-i} for all $i = 0, 1, \dots, m$ and*

$$J_m \subseteq J_{m-1} \subseteq \dots \subseteq J_1 \subseteq J_0,$$

where $J_m = \{0\}$, $J_{m-1} = \{0, c_{m-1}\}$, $J_1 = O_m(\mathbb{F}_2)$ and $J_0 = S_m(\mathbb{F}_2)$.

As a consequence of the proof of Proposition 1, we can write every element of $S_m(\mathbb{F}_2)$ in a certain form.

Corollary 1. *Let $c_i \in J_i \setminus J_{i+1}$ for $i = 0, 2, \dots, m-1$ with $J_0 = S_m(\mathbb{F}_2)$. Then any element of $S_m(\mathbb{F}_2)$ can be written in the form*

$$\beta_0 c_0 + \beta_1 c_1 + \dots + \beta_{m-1} c_{m-1},$$

where $\beta_i \in \mathbb{F}_2$.

2.2 CODES OVER $S_m(\mathbb{F}_2)$

A **(linear)** S_m**-code** of length n is a one-sided $S_m(\mathbb{F}_2)$-submodule of $S_m(\mathbb{F}_2)^n$. Two S_m-codes are **permutation equivalent** if there is a permutation of coordinates that maps one to the other.

The number of nonzero coordinates of a vector $\mathbf{x} \in S_m(\mathbb{F}_2)^n$ is called its **(Hamming) weight** denoted by $wt(\mathbf{x})$. The **(Hamming) distance** $d(\mathbf{x}, \mathbf{y})$ between two vectors $\mathbf{x}, \mathbf{y} \in S_m(\mathbb{F}_2)^n$ is defined as $d(\mathbf{x}, \mathbf{y}) = wt(\mathbf{x}-\mathbf{y})$. The **minimum distance** of an S_m-code $\mathcal{C}$ is

$$\begin{aligned} d(\mathcal{C}) &= \min\{d(\mathbf{x}, \mathbf{y}) | \mathbf{x}, \mathbf{y} \in \mathcal{C}, \mathbf{x} \neq \mathbf{y}\} \\ &= \min\{wt(\mathbf{c}) \mid \mathbf{c} \in \mathcal{C}, \mathbf{c} \neq \mathbf{0}\}. \end{aligned}$$

We endow $S_m(\mathbb{F}_2)^n$ with the usual inner product

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

where $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in S_m(\mathbb{F}_2)^n$. Let C be an S_m-code. The **right dual** of C is the right module defined as

$$C^{\perp_R} = \{\mathbf{y} \in S_m(\mathbb{F}_2)^n \,|\, \forall \mathbf{x} \in C, \mathbf{x} \cdot \mathbf{y} = 0\},$$

and the **left dual** of C is the left module defined as

$$C^{\perp_L} = \{\mathbf{y} \in S_m(\mathbb{F}_2)^n \,|\, \forall \mathbf{x} \in C, \mathbf{y} \cdot \mathbf{x} = 0\}.$$

The **two-sided dual** of C, denoted by $C^{\perp}$ is defined as $C^{\perp} = C^{\perp_R} \cap C^{\perp_L}$. A code is **left self-dual** (resp. right self-dual) if it is equal to its left dual, i.e., $C^{\perp_L} = C$ (resp. right dual, i.e., $C^{\perp_R} = C$). A code C is **self-dual** if $C = C^{\perp}$ and **self-orthogonal** if $C \subseteq C^{\perp}$.

An S_m code C of length n is **left nice** (resp. **right nice**) if $|C|\left|C^{\perp_L}\right| = 2^{mn}$ (resp. $|C|\left|C^{\perp_R}\right| = 2^{mn}$). Moreover, it is called **quasi self-dual** (QSD) if it is self-orthogonal and of size $2^{\frac{mn}{2}}$. A quasi self-dual code with all Hamming weights even is called a **Type IV** code.

Define the map of reduction modulo $O_m(\mathbb{F}_2)$ as the map $\alpha : S_m(\mathbb{F}_2) \to \mathbb{F}_2$ given by $\alpha((a_1, a_2, \ldots, a_m)) = \sum_{i=1}^{m} a_i$. This map can be extended naturally to a map from $S_m(\mathbb{F}_2)^n$ to $\mathbb{F}_2^n$. For an S_m-code C, we associate two binary codes:

1. the **residue code** defined by $res(C) = \{\alpha(\mathbf{y}) \,|\, \mathbf{y} \in C\}$, and
2. the i^{th} **torsion code** for $i \in \{1, 2, \ldots, m-1\}$ defined by

$$tor_i(C) = \{\mathbf{x} \in \mathbb{F}_2^n \,|\, c_i\mathbf{x} \in C\},$$

where $c_0, c_1, \ldots, c_{m-1}$ are fixed such that $c_0 \in S_m(\mathbb{F}_2) \setminus O_m(\mathbb{F}_2)$, $c_i \in O_m(\mathbb{F}_2)$, $i \neq 0$.

Lemma 1. *Let C be an S_m-code. Then every codeword $\mathbf{c} \in C$ can be written as*

$$\mathbf{c} = c_0\mathbf{x}_0 + c_1\mathbf{x}_1 + \ldots + c_{m-1}\mathbf{x}_{m-1},$$

where $\mathbf{x}_0 \in res(C)$ and $\mathbf{x}_i \in \mathbb{F}_2^n$. Moreover, $res(C) \subseteq tor_i(C)$ for $1 \leq i \leq m-2$.

3 SELF-ORTHOGOGNAL AND QSD S_m-CODES

We start with a generalized construction of S_m-codes.

Theorem 2. *Let B_i's be linear codes over $\mathbb{F}_2$ such that $B_0 \subseteq B_i \subseteq B_0^{\perp}$ for $0 \leq i \leq m-1$, where B_0 is self-orthogonal binary code of length n, and $|B_i| = 2^{r_i}$ such that $r_0 + r_1 + \ldots + r_{m-1} = \frac{mn}{2}$. The code C defined by*

$$C = c_0B_0 + c_1B_1 + \ldots + c_{m-2}B_{m-2} + c_{m-1}B_{m-1},$$

is a quasi self-dual code. Its residue code is $res(C) = B_0$ and torsion codes $tor_i(C) = B_i$.

Thus, we can write an S_m-code as a direct sum as follows.

Corollary 2. *If C is a linear code over $S_m(\mathbb{F}_2)$, then*

$$C = c_0B_0 \oplus c_1B_1 \oplus \cdots \oplus c_{m-1}B_{m-1},$$

where $B_0 = res(C)$ and $B_i = tor_i(C)$ for $i = 1, 2, \ldots, m-1$.

Note that we can choose the r_i's such that $r_0 \leq r_{i-1} \leq r_i$ for all $i = 1, 2, \ldots, m-1$.

Corollary 3. *If B_i are binary codes for $i = 0, 1, \ldots, m-1$ such that $B_0 \subseteq B_i$ for all i, then there exist an S_m-code C with residue code B_0 and $tor_i(C) = B_i$. Furthermore, if B_0 is self-orthogonal and $B_i \subseteq B_0^{\perp}$ for all i, then C is self-orthogonal. Moreover, $r_0 + r_1 + \ldots + r_{m-1} = \frac{mn}{2}$ where $|B_i| = 2^{r_i}$ for $0 \leq i \leq m-1$ then C is quasi self-dual code.*

The next result characterizes the residue and torsion codes of self-orthogonal S_m-codes.

Lemma 2. *For all self-orthogonal S_m-linear codes C we have*

1. *$res(C) \subseteq res(C)^{\perp}$;*
2. *$tor_i(C) \subseteq res(C)^{\perp}$;*
3. *$tor_{m-1}(C) = res(C)^{\perp}$ if C is QSD and the sequence $r_0, r_1, r_2, \ldots, r_{m-1}$ is an arithmetic progression.*

Corollary 4. *Let C be an S_m-code of length n. Then C is QSD if and only if $tor_i(C) \subseteq res(C)^{\perp}$ for all i and $r_0 + \ldots + r_{m-1} = \frac{mn}{2}$.*

Theorem 3. *Let C be an S_m-code of order n such that C is QSD and m is even. If there exists $l \in \mathbb{Z}$ such that the sequence $r_{\frac{m}{2}}, \ldots, r_{m-1}$ is the same sequence as $r_0 + l, \ldots, r_{\frac{m}{2}-1} + l$ and $r_{\frac{m}{2}-1} + r_{\frac{m}{2}} = n$, then $tor_{m-1}(C) = res(C)^{\perp}$.*

We have an analog of Lemma 2 for QSD S_m-codes.

Theorem 4. *For all quasi self-dual S_m-linear codes C we have*

1. $res(C) \subseteq res(C)^{\perp}$;
2. $tor_{m-1}(C) \subseteq res(C)^{\perp}$ (if $m = 2$, $tor_{m-1}(C) = res(C)^{\perp}$);
3. if C is of type $\{k_0, \ldots, k_{m-1}\}$, then

$$mk_0 + (m-1)k_1 + \ldots + 2k_{m-2} + k_{m-1} = \frac{mn}{2}.$$

Moreover, if $m \geq 3$, $res(C)$ is self-dual if and only if C is Type IV.

As a consequence, we have the following construction of Type IV codes.

Corollary 5. *If $C = c_0B + c_1B + \ldots + c_{m-1}B$, such that B is binary self-dual code, then C is a Type IV code.*

Finally, we end this section with the general notion of duality of S_m-codes.

Theorem 5. *If C is an S_m-code, then the following hold.*

1. $res(C^{\perp_L}) = tor_i(C^{\perp_L}) = res(C)^{\perp}$ for all $i = 1, 2, \ldots, m-1$
2. $res(C^{\perp_R}) = \bigcap_{i=1}^{m-1} tor_i(C)^{\perp}$
3. $tor_i(C^{\perp_R}) = \mathbb{F}_2^n$ for all $i = 1, 2, \ldots, m-1$

We illustrate all these results in the following example.

Example 1. *Let* $C = c_0 \begin{pmatrix} 0 & 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *Note that* $|C| = 2^3$ *and* $res(C) \subseteq tor_i(C) \subseteq res(C)^{\perp}$ *for* $i = 1, 2$ *which means C is quasi self-dual. Observe that*

$$C^{\perp_R} = c_0 \begin{pmatrix} 0 & 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

since $tor_1(C)^{\perp} = \begin{pmatrix} 1 & 1 \end{pmatrix}^{\perp} = \begin{pmatrix} 1 & 1 \end{pmatrix}$ *and* $tor_2(C)^{\perp} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{\perp} = \begin{pmatrix} 0 & 0 \end{pmatrix}$. *Thus,*

$$res(C^{\perp_R}) = tor_1(C) \cap tor_2(C) = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

and we have

$$C^{\perp_L} = c_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

since $res(C^{\perp_L}) = tor_i(C^{\perp_L}) = res(C)^{\perp} = \begin{pmatrix} 0 & 0 \end{pmatrix}^{\perp} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *Therefore,*

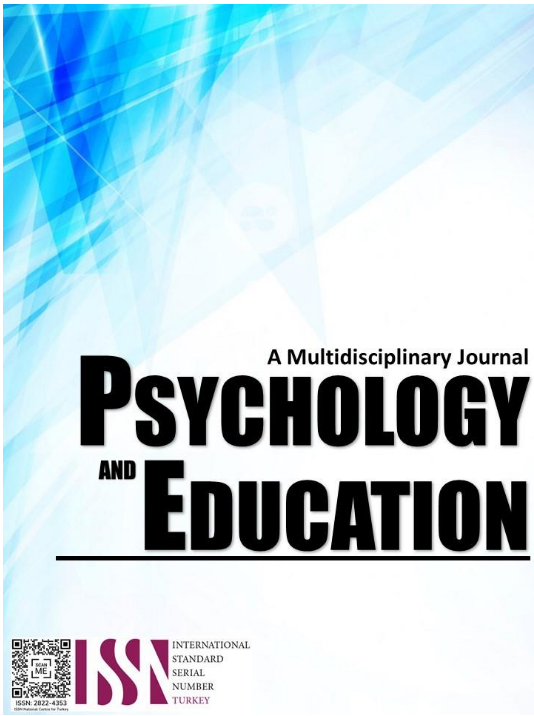
$$C^{\perp} = C^{\perp_R} \cap C^{\perp_L} = c_0 \begin{pmatrix} 0 & 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which means $|C| \cdot |C^{\perp}| = 2^3 \cdot 2^4 = 2^7 \neq 2^6$ *and hence, C is not nice, that is, C is not self-dual.*

3.1 CONCLUSION

The ring $S_m(\mathbb{F}_2)$ is a relatively new ring, which may generalize some known rings. More properties of this ring needs to be explored, especially its application to coding theory and other fields. Future work in codes over this ring includes formulation of more examples for longer length and larger finite fields or other rings in the list of [?]. A complete classification of self-orthogonal, self-dual and QSD S_n-codes for some n will also be valuable work in the future. This can be accomplished using a mass formula, similar to what was done in other rings.

HEURISTIC APPROACH TO NURSING STUDENTS' ACHIEVEMENT TEST SCORES



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Heuristic Approach to Nursing Students' Achievement Test Scores

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Abstract

The study compared various approaches to the teaching of science. The evaluation of performance incorporates teaching and learning concepts. A random selection yielded 82 students of comparable academic standing. Both the pre-test and post-test groups were given instruction that was activity-based. The fact that the control group did better than the experimental group. It demonstrates that the conventional approach is the most effective way to instruct cellular respiration. Hence, the results of the traditional method were significantly better than those of the heuristic one. It has been shown to improve student achievement when taught in a conventional manner. Because males performed better than females, it can be concluded that gender and instructional methods in cellular respiration have no bearing on one another. The paper suggests that conventional approaches to education should be utilized in the classroom more frequently. This is especially the case if the method is able to pique the interest of male students. According to the result of the study, the advantage of heuristics is not restricted in any way because females perform well on heuristic achievement assessments.

Keywords: *conventional method, heuristic method, achievement scores, male, female, cellular respiration*

Introduction

The instructional process affords students the chance to learn new concepts, skills, and procedures. Without instruction, you will not learn any knowledge. There is more to teaching than simply transmitting one's knowledge. It requires educating them on matters they are incorrectly aware of as well as those they are unaware of. Education in the sciences involves the systematic learning of knowledge, with a focus on quantitative study and empirical underpinnings. For the progress of innovation in higher education, the cultivation of inventive talent is crucial. Curriculum, teaching technique, teaching topic, and evaluation methods are all included in the term "teaching system." Heuristic education promotes creative thinking and allows students to improve their skills, which is advantageous for experimental education Heuristic Teaching Method on Innovative Talents Cultivation of Electrical Engineering (2013) and (Zhou 2011).

The scientific approach to education is referred to as heuristics. It accomplishes it in a way that encourages original ideas in s while maintaining educational standards. The application of heuristics in the classroom is advantageous for both students and instructors. A heuristic education seeks to actively engage students in educational activities while simultaneously fostering subjectivity, optimistic thinking, problem-solving skills, and a passion for learning.

It is astounding how attentive the students are during lectures. Occasionally, heuristics are overlooked and it may convey the appearance that the students are uncertain about the answer, do not know the solution, or would know the answer but did not comprehend the elicitation. In order for heuristic teaching to have a refining effect, heuristic training participants must exert greater effort. Every student is expected to participate in both pre-learning activities and classroom participation.

The success of the educational system depends on teachers. Based on their learning objectives, instructors of leadership employ active learning strategies. In addition to imparting knowledge from the textbook, they manage classroom order. Control, evaluation, organization, encouragement, participation, serving as a resource, tutoring, observation, execution, and assistance are all required of the modern educator. Teachers must possess a high level of self-control due to the fact that they assume a variety of roles throughout education. There are four actions that must be completed in order to implement the heuristic technique of instruction.

The study compared several distinct approaches to science education. The core of the performance assessment system is teaching and learning philosophies. There were eighty students with the same amount of educational experience who were selected at random. The participants in the treatment group received standard activity-based training during both the pre-test and post-test stages of this investigation. The fact that the performance of the treatment group exceeded that of the control group suggests that the heuristic approach is the most effective method for teaching cellular respiration.

Research Questions

1. What impact does the heuristic teaching approach have on students' overall cellular respiration achievement scores?
2. How do the students' mean accomplishment scores for men and women differ as a result of the heuristic approach to education?
3. What impact does the gender and approach relationship have on the average student accomplishment scores?

Methodology

A quasi-experimental research design was used in this research paper. A Pretest non-equivalent control group design, specifically. The study was divided into two parts using a straightforward random sample methodology. One section was randomly assigned to the treatment group, and the other was randomly assigned to the control group. The 25 items multiple-choice questionnaire was pilot tested on 16 students who were not research respondents' with a .80 reliability index. These items were chosen from the Krebs cycle, electron transport chain, and glycolysis subtopics. The instrument was put through the face and content validation using a conventional test procedure. The Kuder-Richardson method was used to assess the dependability of the accomplishment

There were two educational programs were used in this study. The second technique is instructive, while the first is heuristic-based. The heuristic approach and the standard package are identical in terms of content, core educational goals, and evaluation methods. The researcher did not select treatment and control groups from the same school to ensure that the pupils in the two groups did not mix. This was done to lessen the possibility of a John Henry effect and to avoid mistakes brought on by interactions and idea-sharing between research participants from the two groups.

Results

Table 1. *Comparing the heristic and conventional group on post-test score*

Strategy	*Mean*	*N*	*Std. Deviation*
Heuristic Method	24.20	40	2.42
Conventional Method	24.64	42	1.24
Total	24.42	82	1.91

According to table 1, students who received instruction using the traditional method had a mean score of 24.64, whereas those who received instruction using the heuristic method had a mean score of 24.20. This demonstrates that the heuristic strategy does not support student achievement as effectively as the traditional method does in fostering students' achievement.

Table 2. *mean accomplishment scores for men and women differ as a result of the heuristic approach to education*

Gender	*Mean*	*N*	*Std. Deviation*
Female	2.00	32	.00
Male	2.00	10	.00
Total	2.00	42	.00

Table 2 shows the male and female students taught in the heuristic method with cellular respiration concepts earned a mean score of 2.00 and a standard deviation of.00. As a result, men and women fared equally well.

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Table 3. *gender and method relationship have on the average student accomplishment scores*

		Gender	*Test Scores*	*Strategy*
Gender	Pearson Correlation	1	.103	-.042
	Sig. (2-tailed)		.355	.706
	N	82	82	82
Test Scores	Pearson Correlation	.103	1	.116
	Sig. (2-tailed)	.355		.297
	N	82	82	82
Strategy	Pearson Correlation	-.042	.116	1
	Sig. (2-tailed)	.706	.297	
	N	82	82	82

Although there is a significant correlation between teaching methods and gender (0.7), there is only a weak correlation between teaching methods and student performance on tests (Table 3). (0.29). It appears from this that there is no connection between gender, an achievement proven to be extreme, and the approach that is taken in the classroom.

Table 4. *students overall scores on cellular respiration and their gender and teaching method*

		Gender	*Test Scores*	*Strategy*
Gender	Pearson Correlation	1	.103	-.042
	Sig. (2-tailed)		.355	.706
	N	82	82	82
Test Scores	Pearson Correlation	.103	1	.116
	Sig. (2-tailed)	.355		.297
	N	82	82	82
Strategy	Pearson Correlation	-.042	.116	1
	Sig. (2-tailed)	.706	.297	
	N	82	82	82

Table 4 displays the link between students' overall scores on cellular respiration and their gender and teaching method. The ANCOVA table for hypothesis 1 indicates that the F-cal (1.18), at a significance level of 0.05, is less than the critical value (.27). The decision rule is to reject the null hypothesis when the calculated value exceeds the critical value with a predetermined probability threshold. Given that the estimated value is less than the essential value, the null hypothesis must hold. Therefore, the researcher concludes that there is no statistically significant difference between the average test results of students who learnt about cellular respiration using the heuristic method and those who learned about it using the conventional way.

Table 4 displays that the two-way interaction F-value is 1234.77, while the critical value for hypothesis 3 at the 95% confidence level is.00. Based on the decision rule, the researcher maintains the null hypothesis and concludes that there is a substantial difference and significant interaction between gender and teaching style in terms of how effectively students learn about cellular respiration.

Table 5. *gender relationship with students' average success scores*

Source		*Type III Sum of Squares*	*df*	*Mean Square*	*F*	*Sig.*
Intercept	Hypothesis	3048.74	1	3048.74	608.15	.00
	Error	76.35	15.23	5.013[a]		
Strategy	Hypothesis	1.18	1	1.18	.52	.47
	Error	158.15	70	2.259[b]		
Gender	Hypothesis	4.59	1	4.59	1.67	.28
	Error	8.40	3.05	2.75[c]		
Pretest	Hypothesis	85.89	7	12.27	4.41	.13
	Error	7.78	2.80	2.77[d]		
Gender * pretest	Hypothesis	5.80	2	2.90	1.28	.28
	Error	158.15	70	2.25[b]		

Table 5 demonstrates that the value of F-cal (1.67), which was calculated using an alpha level of 0.05, is more than the critical value. The estimated number is greater than the critical value at the alpha level that was specified, which means that the null hypothesis is invalid. As a consequence of this, the researcher concludes that the null hypothesis should not be accepted and draws the conclusion that the mean achievement scores of male and female students who were taught cellular respiration using the heuristic method are statistically significantly different from one another.

Discussion

The results of this study indicated that students who were taught cellular respiration using the conventional way did better than those who were taught using the heuristic method. The conventional group's achievement results were attributable to the conceived science being clear and the concepts being connected. Thus, the results of this study contrasted those of Abonyi and Umeh (2014), who found that the heuristic approach is superior to the conventional way and that there is no interaction between genders and linear algebra student achievement. Not all new innovations in teaching and learning increase and attract student learning, inspire, minimize the abstract nature of the subject, and facilitate recollection of taught material, according to the findings of this study.

Using the heuristic method, there was no statistically significant difference between the mean achievement scores of male and female students. The study's heuristic results indicated that there is a substantial relationship between the method and gender in terms of male and female performance in cellular respiration learning. The conventional way indicates that promoting high success in boys is effective. The conventional method generally assumes that different learners with different characteristics learned in the direct teaching-learning process and that the instructional method maximizes the learning outcomes of an instructional method for males since the heuristic method could be effective for a group of male students.

Conclusion

The results of an examination examining the influence that using a heuristic approach has on the level of performance achieved by students studying cellular respiration. The findings demonstrated that the traditional approach was more successful than the heuristic one. As a result, it encourages student achievement. As a consequence of this, males performed much better than females when the standard method was utilized, and there is no evidence to suggest that there is a substantial link between teaching strategy and gender in cellular respiration.

According to the findings of the study, the traditional method of teaching should be utilized more frequently in the classroom. This is particularly the case if the method can attract, excite, and hold the attention of male pupils. As a result of the research, it was determined that the application of heuristics was not restricted because females learned and did well on heuristic accomplishment assessments.

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More on Perfect Roman Domination in Graphs

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Abstract. A *perfect Roman dominating function* on a graph $G = (V(G), E(G))$ is a function $f : V(G) \to \{0, 1, 2\}$ for which each $u \in V(G)$ with $f(u) = 0$ is adjacent to exactly one vertex $v \in V(G)$ with $f(v) = 2$. The *weight* of a perfect Roman dominating function f is the value $\omega_G(f) = \sum_{v\in V(G)} f(v)$. The *perfect Roman domination number* of G is the minimum weight of a perfect Roman dominating function on G. In this paper, we study the perfect Roman domination numbers of graphs under some binary operations.

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Key Words and Phrases: Roman dominating function, perfect Roman dominating function, Roman domination number, perfect Roman domination number

1. Introduction

Throughout this paper, all graphs considered are finite, simple and undirected. Let $G = (V(G), E(G)$ be a graph. The sets $V(G)$ and $E(G)$ are the *vertex set* and *edge set*, respectively, of G. For $S \subseteq V(G)$, $|S|$ is the cardinality of S. In particular, $|V(G)|$ is called the *order* of G. For notation and terminology not given here, see [5].

Vertices u and v of G are *neighbors* if $uv \in E(G)$. The *open neighborhood* of v refers to the set $N_G(v)$ consisting of all neighbors of v. The *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v, denoted $deg_G(v)$, refers to the value $|N_G(v)|$, and we define $\Delta(G) = \max\{deg_G(v) : v \in V(G)\}$. Vertex v is an *endvertex* if $deg_G(v) = 1$, and $End(G)$ is the set of all endvertices of G. Vertex v is an *isolated vertex* if $deg_G(v) = 0$. We denote by $Iso(G)$ the set of all isolated vertices of G. For $S \subseteq V(G)$, $N_G(S) = \cup_{v\in S} N_G(v)$, and $N_G[S] = S \cup N_G(S)$.

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Let G and H be graphs with disjoint vertex sets. The *disjoint union* of G and H is the graph $G \cup H$ with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. The *join* of G and H is the graph $G + H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* of G and H is the graph $G \circ H$ obtained by taking one copy of G and $|V(G)|$ copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H. The *edge corona* of G and H is the graph $G \diamond H$ obtained by taking one copy of G and $|E(G)|$ copies of H and joining each of the end vertices u and v of each edge uv of G to every vertex of the copy H^{uv} of H. The *composition* $G[H]$ of G and H is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u,v)(u',v') \in E(G[H])$ if and only if either $uu' \in E(G)$ or $u = u'$ and $vv' \in E(H)$. The *complementary prism*, denoted $G\overline{G}$, is the graph formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. For the complementary prism, $V(G\overline{G}) = V(G) \cup V(\overline{G})$ and $E(G\overline{G}) = E(G) \cup E(\overline{G}) \cup \{v\overline{v} : v \in V(G)\}$, where $\overline{v}$ is the vertex in $\overline{G}$ corresponding to $v \in V(G)$ in the perfect matching.

A subset $S \subseteq V(G)$ is a *dominating set* of G if $N_G[S] = V(G)$. The minimum cardinality of a dominating set is the *domination number* of G, denoted by $\gamma(G)$. For more details and results on domination number, we refer to [4, 9–11, 13]. In particular, if $\gamma(G) = 1$ and $N_G[v] = V(G)$, then v is said to be a *dominating vertex* of G. In this case, $Dom(G)$ denotes the set of all dominating vertices of G. Any dominating set of G of cardinality $\gamma(G)$ is called *γ-set* of G.

A dominating set S of G is a *perfect dominating set* if for every $v \in V(G) \setminus S$, there exists exactly one $u \in S$ for which $uv \in E(G)$ [16]. The minimum cardinality of a perfect dominating set is the *perfect domination number* of G, which is denoted by $\gamma^P(G)$. Since perfect dominating sets are dominating sets, $\gamma(G) \leq \gamma^P(G)$ for any graph G.

A *Roman dominating function* on G is a function $f : V(G) \to \{0,1,2\}$ satisfying the condition that for each $u \in V(G)$ for which $f(u) = 0$, there exists $v \in V(G)$ such that $f(v) = 2$ and $uv \in E(G)$. The *weight* of f is the value $\omega_G(f) = \sum_{v \in V(G)} f(v)$. The *Roman domination number* of G, denoted by $\gamma_R(G)$, is the minimum weight of a function f on G. We refer to [2, 3, 7, 8, 12, 17, 18] for the history, introduction, importance and for some of the recent developments of the study of Roman domination in graphs.

Customarily, we write $f = (V_0, V_1, V_2)$ for a Roman dominating function f on G, where $V_k = \{v \in V(G) : f(v) = k\}$. With this convention, $\omega_G(f) = |V_1| + 2|V_2|$ and $V_1 \cup V_2$ is a dominating set of G. In [8], it is known that for any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

A *perfect Roman dominating function* (or *PRD-function*) on G is a Roman domination function $f = (V_0, V_1, V_2)$ on G such that for each $u \in V_0$ there exists exactly one $v \in V_2$ for which $uv \in E(G)$. In other words, a PRD-function on G is a colouring of the vertices of G using colours 0, 1 and 2 such that each vertex coloured 0 is adjacent to exactly one vertex coloured 2. The *perfect Roman domination number* of G, denoted by $\gamma_R^P(G)$, is the minimum weight of a PRD-function on G. A PRD-function f with $\omega_G(f) = \gamma_R^P(G)$ is called *γ_R^P-function* of G.

The perfect Roman domination, a variation of the Roman domination, was introduced and first investigated in 2018 by Henning et al. [15], particularly in trees. It is further studied in [14] for regular graphs. More recent studies on the concept include [1, 19, 20].

In this present paper, we continue the study of perfect Roman domination, specifically on the join, corona, complementary prism, edge corona and composition of graphs.

The following bounds are established in the referred articles above.

Theorem 1.1. (i)[15] *If T is a tree of order $n \geq 3$, then $\gamma_R^P(T) \leq \frac{4}{5}n$;*

(ii) [14] *If G is a k-regular graph of order n with $k \geq 4$, then $\gamma_R^P(G) \leq \left(\frac{k^2+k+3}{k^2+3k+1}\right)n$;*

(iii) [19] *If G is a graph of order n, then $\gamma_R^P(G) \leq n+1-\Delta(G)$.*

(iv) [19] *For paths P_n and cycles C_n on $n \geq 3$ vertices, $\gamma_R^P(P_n) = \gamma_R^P(C_n) = \lceil \frac{2n}{3} \rceil$.*

For convenience, we adapt the symbol $PRD(G)$ to denote the set of all perfect Roman dominating functions on the graph G.

2. Results

The following proposition plays an important role in proving the desired results.

Proposition 2.1. *If $f = (V_0, V_1, V_2)$ is a γ_R^P-function of G, then $|N_G(v) \cap V_2| \neq 1$ for each $v \in V_1$.*

Proof: Suppose that there exists $v \in V_1$ for which $|N_G(v) \cap V_2| = 1$. Consider, in particular, the function $f^* = (V_0^*, V_1^*, V_2^*)$ given by $f^*(v) = 0$ and $f^*(x) = f(x)$ for all $x \neq v$. We have $f^* \in PRD(G)$ with $V_0^* = V_0 \cup \{v\}$, $V_1^* = V_1 \setminus \{v\}$ and $V_2^* = V_2$. Thus, $\omega_G(f^*) = \gamma_R^P(G) - 1$, a contradiction. ∎

Proposition 2.2. *For a nontrivial connected graph G of order n,*

$$\max\{2, \gamma(G)\} \leq \gamma_R^P(G) \leq \min\{n+1-\Delta(G), 2\gamma^P(G)\}.$$

Proof: Since a perfect Roman domination is a Roman domination, $\gamma(G) \leq \gamma_R^P(G)$. Let $f = (V_0, V_1, V_2)$ be a γ_R^P-function of G. If $V_0 = \varnothing$, then $\gamma_R^P(G) = n \geq 2$. On the other hand, if $V_0 \neq \varnothing$, then $V_2 \neq \varnothing$ so that $\gamma_R^P(G) \geq 2|V_2| \geq 2$.

By Theorem 1.1(iii), $\gamma_R^P(G) \leq n+1-\Delta(G)$. Now, let $S \subseteq V(G)$ be a γ^P-set of G. Then $f = (V_0, V_1, V_2) \in PRD(G)$, where $V_0 = V(G) \setminus S$, $V_1 = \varnothing$ and $V_2 = S$. Therefore, $\gamma_R^P(G) \leq 2|S| = 2\gamma^P(G)$. ∎

Observe that $\gamma_R^P(C_k) = 4 = k+1-\Delta(C_k) < 2\gamma^P(C_k)$ for $k = 5$ and $\gamma_R^P(C_{3n}) = 2n = 2\gamma^P(C_{3n}) < (3n+1) - \Delta(C_{3n})$ for all $n \geq 2$. Therefore, the upper bound of the inequality in Proposition 2.2 is sharp and may be determined by exactly one of $n+1-\Delta(G)$ and $2\gamma^P(G)$. The inequality, however, can also be strict. To see this, note that $\gamma_R^P(C_7) = 5 < \min\{(7+1) - \Delta(C_7), 2\gamma^P(C_7)\}$.

Corollary 2.3. *Let G be a connected graph of order $n \geq 2$. Then*

(i) [19] $\gamma_R^P(G) = 2$ *if and only if* $\gamma(G) = 1$.

(*ii*) $\gamma_R^P(G) = n$ *if and only if* $n = 2$.

(*iii*) [19] $\gamma_R^P(G) = 3$ *if and only if* $\Delta(G) = n - 2$.

(*iv*) *If* G *is the complete multipartite graph* $K_{r_1,r_2,\ldots,r_m}$*, where* $2 \leq r_1 \leq r_2 \leq \ldots \leq r_m$*, then*

$$\gamma_R^P(G) = \begin{cases} \min\{r_1+1, 4\}, & \text{if } m = 2; \\ r_1 + 1, & \text{if } m \geq 3. \end{cases}$$

Proof: Clearly, if $\gamma(G) = 1$, then $\gamma^P(G) = 1$ and the inequalities in Proposition 2.2 imply that $\gamma_R^P(G) = 2$. Now, suppose that $\gamma_R^P(G) = 2$, and let $f = (V_0, V_1, V_2)$ be a γ_R^P-function of G. If $V_2 = \varnothing$, then $V(G) = V_1$ and $\gamma_R^P(G) = n = 2$. Since G is connected, $G = P_2$ and $\gamma(G) = 1$. If $V_2 \neq \varnothing$, then $V_1 = \varnothing$ and $V_2 = \{v\}$ with $N_G[v] = V(G)$. This means that $\gamma(G) = 1$. This proves (*i*).

If $n = 2$, then $G = P_2$ and $\gamma_R^P(G) = 2 = n$. Conversely, suppose that $n \geq 3$. Pick $v \in V(G)$ such that $deg_G(v) = \Delta(G) \geq 2$. Define on G

$$f(x) = \begin{cases} 2, & \text{if } x = v; \\ 0, & \text{if } x \in N_G(v); \\ 1, & \text{else.} \end{cases}$$

Then $f \in PRD(G)$ and $\omega(f) = n - (\Delta(G) - 1) < n$, a contradiction. Thus, if $\gamma_R^P(G) = n$, then $n = 2$. We have proved (*ii*).

If $\Delta(G) = n - 2$, then Proposition 2.2 implies that $2 \leq \gamma_R^P(G) \leq 3$. Since $\gamma(G) \geq 2$, $\gamma_R^P(G) = 3$ by (*i*). Conversely, suppose that $\gamma_R^P(G) = 3$. By (*i*), $\gamma(G) \geq 2$ so that $\Delta(G) \leq n - 2$, and by (*ii*), $n \geq 4$. Let $f = (V_0, V_1, V_2)$ be a γ_R^P-function on G. If $V_2 = \varnothing$, then $V_1 = V(G)$ and $\gamma_R^P(G) = n \geq 4$, a contradiction. Thus, $|V_2| = |V_1| = 1$, say $V_1 = \{u\}$ and $V_2 = \{v\}$. This means that $V(G) \setminus \{u, v\} \subseteq V_0$. Further, by Proposition 2.1, $uv \notin E(G)$. Accordingly, $deg_G(v) = n - 2$. Therefore, $\Delta(G) \geq n - 2$. This proves (*iii*).

Suppose that G is the complete multipartite graph described in (*iv*). Then $\Delta(G) = n - r_1$. Suppose first that $m = 2$. Then $\gamma(G) = \gamma^P(G) = 2$. By Proposition 2.2, $\gamma_R^P(G) \leq \min\{r_1 + 1, 4\}$. Also, by (*i*), $\gamma_R^P(G) \geq 3$. If $r_1 = 2$, then $\gamma_R^P(G) = 3 = r_1 + 1$. On the other hand, if $r_1 \geq 3$, then $\gamma_R^P(G) = 4 \geq r_1 + 1$. Now, assume that $m \geq 3$. By (*ii*), $\gamma_R^P(G) < n$. Let $f = (V_0, V_1, V_2)$ be a γ_R^P-function on G. Then $|V_2| = 1$, say $V_2 = \{v\}$. Since f is a γ_R^P-function, $v \in U$, where U is the partite set of G with $|U| = r_1$. More precisely, $f(v) = 2$, $f(x) = 1$ for all $x \in U \setminus \{v\}$ and $f(x) = 0$ for all $x \in V(G) \setminus U$. Thus, $\gamma_R^P(G) = \omega_G(f) = r_1 + 1$. This proves (*iv*). ■

Proposition 2.4. [19] *Let* G_1*,* G_2*,* $\ldots$*,* G_k *be the components of* G*. Then* $\gamma_R^P(G) = \sum_{j=1}^{k} \gamma_R^P(G_j)$.

Proposition 2.4 and Corollary 2.3(*ii*) yield the following corollary.

Corollary 2.5. *Let G be a graph of order n. Then $\gamma_R^P(G) = n$ if and only if $G = \cup_{j=1}^k G_j$, where $G_j \in \{K_1, K_2\}$ for all $j = 1, 2, \ldots, k$.*

Corollary 2.6. *Let G be a graph of order n. Then $\gamma(G) = \gamma_R^P(G)$ if and only if $G = \overline{K_n}$.*

Proof: If $G = \overline{K_n}$, then $\gamma(G) = n$ and by Corollary 2.5, $\gamma_R^P(G) = n$. Conversely, suppose that $\gamma(G) = \gamma_R^P(G)$, and let $f = (V_0, V_1, V_2)$ be a γ_R^P-function of G. Note that if $V_2 \neq \varnothing$, then $\gamma(G) \leq |V_1| + |V_2| < \gamma_R^P(G)$, a contradiction. Thus, $V_2 = V_0 = \varnothing$ and $\gamma_R^P(G) = n$. This means that $\gamma(G) = n$ and, thus, $G = \overline{K_n}$. ■

2.1. On the join of graphs

By Corollary 2.3(i), $\gamma_R^P(G + K_n) = 2$ for all graphs G and for all $n \geq 1$.

The following theorem characterizes all PRD-functions on the join of nontrivial connected graphs.

Theorem 2.7. *Let G and H be any nontrivial connected graphs and $f = (V_0, V_1, V_2)$. Then $f \in PRD(G + H)$ if and only if one of the following holds:*

(i) *$V_2 \subseteq V(G)$ and one of the following holds:*

- (a) *$V_0 \subseteq V(G)$, $V(H) \subseteq V_1$ and $(V_0, V_1 \cap V(G), V_2) \in PRD(G)$;*
- (b) *$V_0 \cap V(H) \neq \varnothing$ and $V_2 = \{v\}$ for which $V_0 \cap V(G) \subseteq N_G(v)$.*

(ii) *$V_2 \subseteq V(H)$ and one of the following holds:*

- (a) *$V_0 \subseteq V(H)$, $V(G) \subseteq V_1$ and $(V_0, V_1 \cap V(H), V_2) \in PRD(H)$;*
- (b) *$V_0 \cap V(G) \neq \varnothing$ and $V_2 = \{v\}$ for which $V_0 \cap V(H) \subseteq N_H(v)$.*

(iii) *$A_1 = V_2 \cap V(G) \neq \varnothing$ and $A_2 = V_2 \cap V(H) \neq \varnothing$ and the following holds:*

- (a) *If $V_0 \cap V(G) \neq \varnothing$, then $|A_2| = 1$ and $(V_0 \cap V(G)) \cap N_G(A_1) = \varnothing$;*
- (b) *If $V_0 \cap V(H) \neq \varnothing$, then $|A_1| = 1$ and $(V_0 \cap V(H)) \cap N_H(A_2) = \varnothing$.*

Proof: Assume that f is a perfect Roman dominating function on $G + H$. We consider three cases:

Case 1: Suppose that $V_2 \subseteq V(G)$. If $V_0 \subseteq V(G)$, then $V(H) \subseteq V_1$ and the restriction $f|_{V(G)} = (V_0, V_1 \cap V(G), V_2)$ of f on G is a perfect dominating function on G. Suppose that $V_0 \cap V(H) \neq \varnothing$. Then, $|V_2| = 1$, say $V_2 = \{v\}$. Necessarily, $V_0 \cap V(G) \subseteq N_G(v)$.

Case 2: Similarly, if $V_2 \subseteq V(H)$, then either (ii)(a) or (ii)(b) holds.

Case 3: Assume that V_2 intersects both $V(G)$ and $V(H)$, and $A_1 = V_2 \cap V(G)$ and $A_2 = V_2 \cap V(H)$. Suppose that $V_0 \cap V(G) \neq \varnothing$, and let $v \in V_0 \cap V(G)$. Since $A_2 \subseteq N_{G+H}(v)$, $|A_2| = 1$ and $v \notin N_G(A_1)$. Since v is arbitrary, (iii)(a) holds. Similarly, (iii)(b) holds.

Conversely, suppose that (i)(a) holds for f, and let $w \in V_0$. Then $w \in V(G)$ and there exists a unique $u \in V_2$ for which $uw \in E(G)$. Since $V(H) \subseteq V_1$, u is unique in $V(G+H)$ for which $uw \in E(G+H)$. This means that $f \in PRD(G+H)$. Suppose that $(i)(b)$ holds for f, and let $w \in V_0$. Whether $w \in V(G)$ or $w \in V(H)$, v is a unique element in V_2 for which $wv \in E(G+H)$. Thus, $f \in PRD(G+H)$. Similarly, if (ii) holds, the same conclusion is attained for f. Suppose now that (iii) holds for f. Let $v \in V_0$. If $v \in V(G)$, then by condition (a), $A_2 = \{u\}$ for some $u \in V(H)$ and $N_{G+H}(v) = \{u\}$. Similarly, if $v \in V(H)$, then $A_1 = \{u\}$ for some $u \in V(G)$ and $N_{G+H}(v) = \{u\}$. Accordingly, $f \in PRD(G+H)$. ■

We now use Theorem 2.7 to prove the following result which is also provided in [19].

Corollary 2.8. [19] *Let Gand H be nontrivial connected graphs of orders m and n, respectively. Then*

$$\gamma_R^P(G+H) = min\{4+\delta(G)+\delta(H), m+1-\Delta(G), n+1-\Delta(H)\}.$$

Proof: Let $\alpha = min\{4+\delta(G)+\delta(H), m+1-\Delta(G), n+1-\Delta(H)\}$. Let $v \in V(G)$ for which $deg_G(v) = \Delta(G)$. Define $f = (V_0, V_1, V_2)$ on $G+H$ by

$$f(x) = \begin{cases} 2, & \text{if } x = v; \\ 0, & \text{if } x \in V(H) \cup N_G(v); \\ 1, & \text{else.} \end{cases}$$

Since f satisfies condition $(i)(b)$ of Proposition 2.7, $f = (V_0, V_1, V_2) \in PRD(G+H)$ with $V_2 = \{v\}$ and $V_1 = V(G) \setminus N_G[v]$. Thus,

$$\begin{aligned} \gamma_R^P(G+H) \leq \omega_{G+H}(f) &= |V(G) \setminus N_G[v]| + 2 \\ &= m+1-\Delta(G). \end{aligned}$$

Similarly, $\gamma_R^P(G+H) \leq n+1-\Delta(H)$.

Now, pick $u \in V(G)$ and $v \in V(H)$ such that $deg_G(u) = \delta(G)$ and $deg_H(v) = \delta(H)$, and define $f = (V_0, V_1, V_2)$ on $G+H$ by

$$f(x) = \begin{cases} 2, & \text{if } x = u, v; \\ 1, & \text{if } x \in N_G(u) \cup N_H(v); \\ 0, & \text{else.} \end{cases}$$

Since f satisfies Proposition 2.7 (iii), $f \in PRD(G+H)$. Since $V_2 = \{u, v\}$ and $V_1 = N_G(u) \cup N_H(v)$,

$$\begin{aligned} \gamma_R^P(G+H) \leq \omega_{G+H}(f) &= |N_G(u) \cup N_H(v)| + 4 \\ &= 4+\delta(G)+\delta(H). \end{aligned}$$

All of the above show that $\gamma_R^P(G+H) \leq \alpha$.

Now, let $f = (V_0, V_1, V_2)$ be a γ_R^P-function of $G+H$. By Corollary 2.3(ii), since $m+n \geq 4$, $V_2 \neq \varnothing$. Assume $A_1 = V_2 \cap V(G) \neq \varnothing$. We consider two cases:

Case 1: Suppose that $A_2 = V_2 \cap V(H) = \varnothing$. If Proposition 2.7$(i)(a)$ holds for f, then

$$\omega_{G+H}(f) \geq n + \gamma_R^P(G) > n \geq n + 1 - \Delta(H) \geq \alpha.$$

On the other hand, if Proposition 2.7$(i)(b)$ holds for f, then

$$\omega_{G+H}(f) \geq 2 + |V(G) \setminus N_G[v]| \geq m + 1 - \Delta(G) \geq \alpha.$$

Case 2: Suppose that $A_2 = V_2 \cap V(H) \neq \varnothing$. If $|A_1| \geq 2$ and $|A_2| \geq 2$, then $V_0 = \varnothing$ and $\gamma_R^P(G+H) > m+n$, which is impossible. Assume that $|A_2| = 1$. We consider two subcases. First, suppose that $|A_1| \geq 2$. Then $V_0 \cap V(H) = \varnothing$, and since f is a γ_R^P-function of $G+H$, $V(G) \setminus N_G[A_1] \subseteq V_0$ (by Proposition 2.1) and $N_G(A_1) \setminus A_1 \subseteq V_1$. This means that $|V_1| \geq |V(H) \setminus V_2| + |N_G(A_1) \setminus A_1|$ so that

$$\omega_{G+H}(f) = (n-1) + |N_G(A_1) \setminus A_1| + 2|V_2| \geq n + 5 > n + 1 - \Delta(H).$$

Finally, suppose that $|A_1| = 1$. Let $A_1 = \{u\}$ and $A_2 = \{v\}$ for some $u \in V(G)$ and $v \in V(H)$. By Proposition 2.1, $f(x) = 0$ for all $x \in V(G+H) \setminus (N_G[u] \cup N_H[v])$. Thus,

$$\omega_{G+H}(f) \geq 2|A_1 \cup A_2| + |N_G(u) \cup N_H(v)| \geq 4 + \delta(G) + \delta(H) \geq \alpha.$$

All cases above imply that $\gamma_R^P(G+H) \geq \alpha$. ■

In particular, if $m \geq n$, then

$$\gamma_R^P(P_m + P_n) = \begin{cases} n-1, & \text{if } n \leq 6; \\ 6, & \text{if } n \geq 7. \end{cases} \quad \text{and} \quad \gamma_R^P(C_m + P_n) = \begin{cases} n-1, & \text{if } n \leq 7; \\ 7, & \text{if } n \geq 8. \end{cases}$$

2.2. On the corona of graphs

Let G and H be connected graphs. Adapting the notation used in [6], for each $v \in V(G)$, H^v denotes that copy of H which is joined with v in $G \circ H$. In case $H = \{x\}$, we write $V(H^v) = \{x^v\}$. Then $V(G+H) = \cup_{v \in V(G)} V(H^v + v)$, where $H^v + v = H^v + \langle v \rangle$.

It is worth noting that $K_1 \circ H = H + K_1$ for any graph H.

Theorem 2.9. *For nontrivial connected graphs G of order n,*

$$\gamma_R^P(G \circ K_1) = \min\{\omega_G(f) + n - |V_2| : f = (V_0, V_1, V_2) \in PRD(G)\}.$$

In particular, $\gamma_R^P(K_n \circ K_1) = n + 1$.

Proof: Write $H = \{x\}$, and put $\alpha = \min\{\omega_G(f) + n - |V_2| : f = (V_0, V_1, V_2) \in PRD(G)\}$. Let $f = (V_0, V_1, V_2) \in PRD(G)$. Define $f^* = (V_0^*, V_1^*, V_2^*)$ on $G \circ K_1$ by

$$f^*(z) = \begin{cases} f(z), & \text{if } z \in V(G); \\ 1, & \text{if } z = x^v \text{ for some } v \in V_0 \cup V_1; \\ 0, & \text{if } z = x^v \text{ for some } v \in V_2. \end{cases}$$

Then $f^* \in PRD(G \circ K_1)$ with $V_0^* = V_0 \cup \{x^v : v \in V_2\}$, $V_1^* = V_1 \cup \{x^v : v \in V_0 \cup V_1\}$ and $V_2^* = V_2$. Moreover,

$$\omega_{G \circ K_1}(f^*) = \omega_G(f) + n - |V_2|.$$

Thus, $\gamma_R^P(G \circ K_1) \leq \alpha$.

Let $f = (V_0, V_1, V_2)$ be a γ_R^P-function on $G \circ K_1$, and let A denote the set of all $u \in V_0 \cap V(G)$ for which $uv \notin E(G)$ for all $v \in V_2 \cap V(G)$. Then for each $u \in A$, $V_2 \cap N_{G \circ K_1}(u) = \{x^u\}$. Define $f^* = (V_0^*, V_1^*, V_2^*)$ on $G \circ K_1$ by

$$f^*(z) = \begin{cases} f(z), & \text{if } z \in V(G) \setminus A; \\ 1, & \text{if } z \in A \cup \{x^u : u \in (V_0 \cup V_1) \cap V(G)\}; \\ 0, & \text{if } z \in \{x^v : \ v \in V_2 \cap V(G)\}. \end{cases}$$

Then $f^* \in PRD(G \circ K_1)$ with $V_0^* = ((V_0 \cap V(G)) \setminus A) \cup \{x^u : u \in V_2 \cap V(G)\}$, $V_1^* = A \cup (V_1 \cap V(G)) \cup \{x^u : u \in (V_0 \cup V_1) \cap V(G)\}$ and $V_2^* = V_2 \cap V(G)$. Observe that $f(u) + f(x^u) = 2 = f^*(u) + f^*(x^u)$ for each $u \in A$, and $f(u) + f(x^u) \geq f^*(u) + f^*(x^u)$ for each $u \in V(G) \setminus A$. Thus,

$$\begin{aligned} \omega_{G \circ K_1}(f) &= \sum_{u \in A} (f(u) + f(x^u)) + \sum_{v \in V(G) \setminus A} (f(u) + f(x^u)) \\ &\geq \sum_{u \in A} (f^*(u) + f^*(x^u)) + \sum_{u \in V(G) \setminus A} (f^*(u) + f^*(x^u)) \\ &= \omega_{G \circ K_1}(f^*). \end{aligned}$$

Since f is a γ_R^P-function, $\omega_{G \circ K_1}(f) = \omega_{G \circ K_1}(f^*)$. Moreover, for each $u \in V_0^* \cap V(G)$, $u \in (V_0 \cap V(G)) \setminus A$ so that there exists a unique $v \in V_2 \cap V(G) = V_2^*$ such that $uv \in E(G)$. This means that the restriction $f^*|_G$ of f^* to G is a perfect Roman dominating function on G. Thus,

$$\begin{aligned} \gamma_R^P(G \circ K_1) = \omega_{G \circ K_1}(f^*) &= \omega_G(f^*|_G) + \sum_{v \in V(G)} f^*(x^v) \\ &= \omega_G(f^*|_G) + |\,(V_0 \cup V_1) \cap V(G)| \\ &= \omega_G(f^*|_G) + n - |V_2^* \cap V(G)| \\ &\geq \alpha. \end{aligned}$$

■

It follows from Theorem 2.9 that for all connected graphs G of order $n \geq 2$,

$$\gamma_R^P(G \circ K_1) \leq \gamma_R^P(G) + n - \lambda,$$

where $\lambda = \max\{|V_2| : (V_0, V_1, V_2) \text{ is a } \gamma_R^P\text{-function on } G\}$, and this bound is sharp. Verify that equality is attained if G is a cycle C_n $(n \geq 3)$, a path P_n $(n \geq 2)$, or any graph with $\gamma(G) = 1$.

Our desired result for more general graphs G and H will follow from the following characterization.

Theorem 2.10. *Let G and H be nontrivial graphs with G connected, and $f = (V_0, V_1, V_2)$. Then $f \in PRD(G \circ H)$ if and only if the following holds:*

(i) For all $v \in V_0 \cap V(G)$ either

(a) $V_2 \cap N_G(v) = \varnothing$ and $V_2 \cap V(H^v) = \{u\}$ with u satisfying $V_0 \cap V(H^v) \subseteq N_{H^v}(u)$; or

(b) $|V_2 \cap N_G(v)| = 1$ and $V(H^v) \subseteq V_1$;

(ii) For all $v \in V_1 \cap V(G)$, the restriction $f|_{H^v}$ of f to H^v is a perfect Roman dominating function on H^v;

(iii) For all $v \in V_2 \cap V(G)$ for which $V_0 \cap V(H^v) \neq \varnothing$, $V_0 \cap N_{H^v}(V_2 \cap V(H^v)) = \varnothing$.

Proof: Assume that $f \in PRD(G \circ H)$. Let $v \in V_0 \cap V(G)$. Then there exists a unique $u \in V_2$ for which $u \in N_{G \circ H}(v) = V(H^v) \cup N_G(v)$. If $V_2 \cap N_G(v) = \varnothing$, then $V_2 \cap V(H^v) = \{u\}$ and $V_0 \cap V(H^v) \subseteq N_{H^v}(u)$. Suppose that $V_2 \cap N_G(v) \neq \varnothing$. Then $|V_2 \cap N_G(v)| = 1$ and $V_2 \cap V(H^v) = \varnothing$. Moreover, if $w \in V_0 \cap V(H^v)$, then there exists a unique $z \in V_2 \cap V(H^v)$ such that $wz \in E(H^v)$. Since $vz \in E(G \circ H)$, this is impossible. Thus, $V(H^v) \subseteq V_1$. This proves (i). Next, let $v \in V_1 \cap V(G)$, and let $w \in V_0 \cap V(H^v)$. Since f is a perfect Roman dominating function, there exists unique $u \in V_2$ for which $uw \in E(G \circ H)$. Since $v \in V_1$, $u \in V_2 \cap V(H^v)$ and $uw \in E(H^v)$. Thus, $f|_{H^v}$ is a perfect Roman dominating function on H^v, and (ii) holds. Statement (iii) is clear.

Conversely, suppose that conditions (i), (ii) and (iii) hold for f, and let $w \in V_0$. Then $w \in V(H^v + v)$ for some $v \in V(G)$. If $w = v$, then by condition (i), $V_2 \cap (V(H^v) \cup N_G(w)) = \{u\}$ for some $u \in V(G \circ H)$. This means that $V_2 \cap N_{G \circ H}(w) = \{u\}$. Suppose that $w \in V(H^v)$. We consider three cases:

Case 1: Suppose that $v \in V_0$. Since $w \in V_0 \cap V(H^v)$, $V(H^v) \nsubseteq V_1$. Thus, by condition (i) there exists $u \in V(H^v)$ for which $V_2 \cap V(H^v) = \{u\}$ and $V_0 \cap V(H^v) \subseteq N_{H^v}(u)$. This means that $V_2 \cap N_{G \circ H}(w) = \{u\}$.

Case 2: Suppose that $v \in V_1$. By condition (ii), there exists a unique $u \in V_2 \cap V(H^v)$ such that $uw \in E(H^v) \subseteq E(G \circ H)$. This implies that $V_2 \cap N_{G \circ H}(w) = \{u\}$.

Case 3: Suppose that $v \in V_2$. Since $w \in V_0 \cap V(H^v)$, condition (iii) implies that $w \notin N_{H^v}(V_2 \cap V(H^v)$. Thus, $V_2 \cap N_{G \circ H}(w) = \{v\}$.

Therefore, f is a perfect Roman dominating function on $V(G \circ H)$. ■

Corollary 2.11. *Let G and H be nontrivial graphs with G connected of order n. Then $\gamma_R^P(G \circ H) = 2n$.*

Proof: By Theorem 2.7, the function $f = (V_0, V_1, V_2)$ defined by $f(x) = 2$ for all $v \in V(G)$, and $f(x) = 0$ else, is a perfect Roman dominating function on $G \circ H$. Thus, $\gamma_R^P(G \circ H) \leq 2n$.

Now, let $f = (V_0, V_1, V_2)$ be a γ_R^P-function on $V(G \circ H)$. Let $v \in V(G)$. Clearly, if $v \in V_2$, then $\sum_{x \in V(H^v + v)} f(x) \geq 2$. If $v \in V_0$, then by Proposition 2.10(i) and since

$|V(H^v| \geq 2$, $\sum_{x\in V(H^v+v)} f(x) \geq 2$. Finally, if $v \in V_1$, then by Proposition 2.10(ii), $\sum_{x\in V(H^v+v)} f(x) > 2$. Therefore,

$$\gamma_R^P(G\circ H) = \omega_{G\circ H}(f) = \sum_{v\in V(G)} \left(\sum_{x\in V(H^v+v)} f(x) \right) \geq 2n.$$

■

2.3. On the complementary prisms

Let $f = (V_0, V_1, V_2) \in PRD(G\overline{G})$. Suppose that for the restriction $f|_{\overline{G}} \notin PRD(\overline{G})$. Then there exists $v \in V(G)$ such that $\overline{v} \in V_0$ and $V_2 \cap N_{G\overline{G}}(\overline{v}) = \{v\}$. Let $u \in V_0 \cap V(G)$. There exists $w \in V(G\overline{G})$ such that $V_2 \cap N_{G\overline{G}}(u) = \{w\}$. If $w = \overline{u}$, then $\overline{uv} \notin E(\overline{G}$, and consequently, $uv \in E(G)$, a contradiction. Thus, $w \in V_2 \cap V(G)$. This proves the following lemma.

Lemma 2.12. *Let G be any graph. If $f \in PRD(G\overline{G})$, then $f|_G \in PRD(G)$ or $f|_{\overline{G}} \in PRD(\overline{G})$.*

Proposition 2.13. *Let G be a graph of order n. Then*

(i) $\gamma(G\overline{G}) < \gamma_R^P(G\overline{G})$;

(ii) $\gamma_R^P(G\overline{G}) = 2$ *if and only if* $n = 1$;

(iii) $\gamma_R^P(G\overline{G}) = 3$ *if and only if* $G \in \{K_2, \overline{K_2}\}$;

(iv) *If* $\gamma(G) = 1$, *then* $\gamma_R^P(G\overline{G}) \leq n+1$ *and equality is attained if* $deg_G(v) \leq 3$ *for all* $v \notin Dom(G)$ *or* $\overline{G}$ *is the disjoint union of* $K_j \in \{K_1, K_2\}$.

Proof: Since $G\overline{G}$ is connected, (i) follows from Corollary 2.6.

If $n = 1$, then $G\overline{G} = K_2$ and $\gamma_R^P(G\overline{G}) = 2$. Suppose that $\gamma_R^P(G\overline{G}) = 2$, and let f be a γ_R^P-function of $G\overline{G}$. By Lemma 2.12, we may assume that $f|_G \in PRD(G)$. If $\omega_G(f|_G) = 1$, then $n = 1$. If $\omega_G(f|_G) = 2$, then $G = \{v\}$ with $f(v) = f|_G(v) = 2$ and $f(\overline{v}) = 0$.

If $G \in \{K_2, \overline{K_2}\}$, then $G\overline{G}$ is isomorphic to P_4. Thus, $\gamma_R^P(G\overline{G}) = 3$. Conversely, suppose that $\gamma_R^P(G\overline{G}) = 3$. By Proposition 2.3($iii$), $\Delta(G\overline{G}) = 2n-2$. Let $v \in V(G\overline{G})$ be such that $deg_{G\overline{G}}(v) = 2n-2$. Without loss of generality, assume that $v \in V(G)$. Since $N_{G\overline{G}}(v) \cap V(\overline{G}) = \{\overline{v}\}$, $deg_G(v) = 2n-3 \leq n-1$. Necessarily, $n \leq 2$. By (ii), $n = 2$ and $G = K_2$.

If $\gamma(G) = 1$, then by Proposition 2.2, $\gamma_R^P(G\overline{G}) \leq n+1$. First, suppose that $deg_G(v) \leq 3$ for all $v \notin Dom(G)$. Let $f = (V_0, V_1, V_2)$ be a γ_R^P-function of $G\overline{G}$. Since $\omega_{G\overline{G}}(f) \leq n+1$, $V_2 \neq \varnothing$. We consider two cases:

Case 1: Suppose that $V_2 \cap V(G) = \varnothing$. If $V(G) \subseteq V_1$, then $V(\overline{G}) \nsubseteq V_0$ so that $\omega_{G\overline{G}}(f) \geq n+1$. Suppose that $V(G) \cap V_0 \neq \varnothing$. Then

$$\begin{aligned}\omega_{G\overline{G}}(f) &= \sum_{w \in V_0 \cap V(G)} f(\overline{w}) + \sum_{w \in V_1 \cap V(G)} (f(w) + f(\overline{w})) \\ &\geq n+1.\end{aligned}$$

Case 2: Assume that $V_2 \cap V(G) \neq \varnothing$. We consider two subcases:

Subcase 2.1: Suppose that V_2 contains a dominating vertex v of G. Since f is a γ_R^P-function, $N_G(v) \cup \{\overline{v}\} \subseteq V_0$. Let $w \in V(G) \setminus \{v\}$. Suppose that $\overline{w} \in V_0$. There exists $u \in V(G)$ such that $N_{\overline{G}}(\overline{w}) \cap V_2 = \{\overline{u}\}$. Since $\overline{wv} \notin E(\overline{G})$, $u \neq v$. Thus, $u \in V_0$ and $v, \overline{u} \in N_{G\overline{G}}(u) \cap V_2$, a contradiction. This means that $f(\overline{w}) \geq 1$. Therefore, $\omega_{G\overline{G}}(f) = 2 + \sum_{w \in V(G)\setminus\{v\}} f(\overline{w}) \geq 2 + n - 1 = n+1$.

Subcase 2.2: Suppose that $V_2 \cap Dom(G) = \varnothing$. Choos $v \in Dom(G)$. Put $A = \{w \in V(G) : f(w) = f(\overline{w}) = 0\}$. If $A = \varnothing$, then $f(w) + f(\overline{w}) \geq 1$ for all $w \in V(G)$ and since $V_2 \cap V(G) \neq \varnothing$, we have $\omega_{G\overline{G}}(f) \geq n+1$. Suppose that $A \neq \varnothing$. Here, we work on two subcases:

Subcase 2.2.1: Suppose that $v \in V_0$. If $f(\overline{v}) = 2$, then $V(G) \cap V_2 = \varnothing$ and so $f(\overline{u}) = 2$ for each $u \in V_0 \cap V(G)$. This implies that $\omega_{G\overline{G}}(f) \geq n+1$. Suppose that $f(\overline{v}) = 1$. Then there exists $u \in V(G)$ such that $V_2 \cap V(G) = \{u\}$. Moreover, for each $w \in A$, $wu \in E(G)$. Since $deg_G(u) \leq 3$ and $uv \in E(G)$, $|A| \leq 2$. Suppose that $A = \{w\}$. There exists $a \in V(G)$ such that $u \neq a$ and $N_{\overline{G}}(\overline{w}) \cap V_2 = \{\overline{a}\}$. Since $\alpha = (f(u) + f(\overline{u})) + (f(w) + f(\overline{w})) + (f(a) + f(\overline{a})) \geq 4$,

$$\omega_{G\overline{G}}(f) = \alpha + \sum_{x \in V(G)\setminus\{u,w,a\}} (f(x) + f(\overline{x})) \geq 4 + (n-4) + 1 = n+1.$$

Now, suppose that $A = \{w, z\}$. There exist $a, b \in V(G)$ such that $\overline{a}, \overline{b} \in V_2$, $\overline{wa}, \overline{zb} \in E(\overline{G})$ and $\overline{a}, \overline{b} \in N_{\overline{G}}(\overline{u})$. Thus, $f(\overline{u}) = f(a) = f(b) = 1$ and whether $a = b$ or $a \neq b$,

$$\alpha = (f(u) + f(\overline{u})) + (f(w) + f(\overline{w})) + (f(z) + f(\overline{z})) + (f(a) + f(\overline{a})) + \big(f(b) + f(\overline{b})\big) \geq 6.$$

Thus,

$$\omega_{G\overline{G}}(f) = \alpha + \sum_{x \in V(G)\setminus\{u,w,z,a,b\}} (f(x) + f(\overline{x})) \geq 6 + (n-6) + 1 = n+1.$$

Subcase 2.2.2: Suppose that $v, \overline{v} \in V_1$. For each $w \in A$, there exist distinct vertices $u, z \in V(G)$ such that $u, \overline{z} \in V_2$, $uw \in E(G)$ and $\overline{wz} \in E(\overline{G})$. Again, for each $u \in V_2 \cap V(G)$, since $deg_G(u) \leq 3$, there can only be at most two vertices $a, b \in A$ for which $ua, ub \in E(G)$. Using similar arguments, if $|A| \leq 2$, then $\omega_{G\overline{G}}(f) \geq n+1$. To proceed, we only have to consider the case where $3 \leq |A| \leq 4$. Other cases follow inductively.

Suppose that $A = \{x, y, w\}$. The only nontrivial scenario is the following: There exist $a, c \in V_2 \cap V(G)$ and $b \in V(G)$ such that $\overline{b} \in V_2$, $ac \notin E(G)$, $wc \in E(G)$, $\{x, y\} \subseteq N_G(a)$, and $\{\overline{x}, \overline{y}, \overline{w}\} \subseteq N_{\overline{G}}(\overline{b})$. Since $\overline{a}\overline{b} \in E(\overline{G})$, $f(\overline{a}) = 1$. Thus,

$$\begin{aligned}
\omega_{G\overline{G}}(f) &= \sum_{u \in \{a,x,y,b,w,c\}} (f(u) + f(\overline{u})) + \sum_{u \in V(G)\setminus\{a,b,c,x,y,w\}} (f(u) + f(\overline{u})) \\
&\geq 7 + (n-7) + 2 \\
&> n+1.
\end{aligned}$$

Finally, suppose that $A = \{x, y, z, w\}$. It is enough to consider only the following nontrivial case: There exist $a, c \in V_2 \cap V(G)$ and $b \in V(G)$ such that $\overline{b} \in V_2$, $ac \notin E(G)$, $\{x, y\} \subseteq N_G(a)$, $\{w, z\} \subseteq N_G(c)$, and $\{\overline{x}, \overline{y}, \overline{z}, \overline{w}\} \subseteq N_{\overline{G}}(\overline{b})$. Since $\overline{a}\overline{b}, \overline{c}\overline{b} \in E(\overline{G})$, $f(\overline{a}) = f(\overline{c}) = 1$. Hence,

$$\begin{aligned}
\omega_{G\overline{G}}(f) &= \sum_{u \in \{a,b,c,x,y,w,z\}} (f(u) + f(\overline{u})) + \sum_{u \in V(G)\setminus\{a,b,c,x,y,w,z\}} (f(u) + f(\overline{u})) \\
&\geq 8 + (n-8) + 2 \\
&> n+1.
\end{aligned}$$

All of the above cases show that $\gamma_R^P(G) = \omega_{G\overline{G}}(f) \geq n+1$.

Next, suppose that $\overline{G}$ is the union of $K_j \in \{K_1, K_2\}$, and let $f = (V_0, V_1, V_2)$ be a γ_R^P-function of $G\overline{G}$. As shown previously, we may assume that $V_2 \cap V(G) \neq \varnothing$, and if V_2 contains a dominating vertex of G, then $\omega_{G\overline{G}}(f) \geq n+1$. Henceforth, we assume that $V_2 \cap Dom(G) = \varnothing$. Pick $v \in Dom(G)$. Then $\overline{v} \in Iso(\overline{G})$. Note that for all $\overline{x} \in Iso(\overline{G})$, $x \notin A = \{w \in V(G) : f(w) = f(\overline{w}) = 0\}$ so that $(f(x) + f(\overline{x})) \geq 1$. Also, for all $x, y \in V(G)$ for which $\overline{x}\overline{y} \in E(\overline{G})$, if $x \in A$, then $\overline{y} \in V_2$ and so $(f(x) + f(\overline{x})) + (f(y) + f(\overline{y})) \geq 2$. Thus, if $v \in V_0$ and $u \in V(G)$ such that $V_2 \cap V(G) = \{u\}$, then

$$\begin{aligned}
\omega_{G\overline{G}}(f) &= (f(u) + f(\overline{u})) + \sum_{\overline{x} \in Iso(\overline{G})} (f(x) + f(\overline{x})) + \\
&\quad \sum_{\overline{x}\overline{y} \in E(\overline{G})} ((f(x) + f(\overline{x})) + (f(y) + f(\overline{y}))) \\
&\geq n+1.
\end{aligned}$$

On the other hand, if $v \in V_1$, then $f(\overline{v}) = 1$ and

$$\begin{aligned}
\omega_{G\overline{G}}(f) &= (f(v) + f(\overline{v})) + \sum_{\overline{x} \in Iso(\overline{G})\setminus\{\overline{v}\}} (f(x) + f(\overline{x})) + \\
&\quad \sum_{\overline{x}\overline{y} \in E(\overline{G})} ((f(x) + f(\overline{x})) + (f(y) + f(\overline{y}))) \\
&\geq n+1.
\end{aligned}$$

Therefore, $\gamma_R^P(G\overline{G}) \geq n+1$. ■

As shown by the graph G in Figure 1, strict inequality may be attained in Proposition 2.13(iv) if we remove the condition that $deg_G(v) \leq 3$ for all nondominating vertices v of G. For such G, $\gamma_R^P(G\overline{G}) = 6 < |V(G)| + 1$.

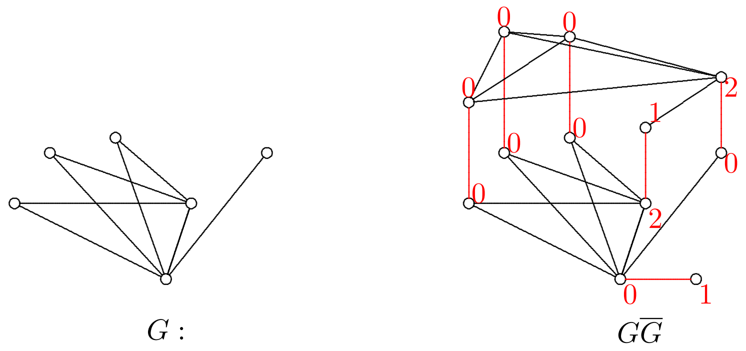

Figure 1: Graph G with $\gamma(G) = 1$ and $\gamma_R^P(G\overline{G}) < |V(G)| + 1$

Pick $G = K_n$. By Proposition 2.13(iv) and Corollary 2.5,

$$\gamma_R^P(G\overline{G}) = 1 + \max\{\gamma_R^P(G), \gamma_R^P(\overline{G})\}.$$

Observe also that if $v \in V(G)$, then $f = (V(G) \setminus \{v\}, \varnothing, \{v\}) \in PRD(G)$ and $\gamma_R^P(G\overline{G}) = \omega_G(f) + n - |V_2|$. The following result shows that these two expressions serve as sharp lower and upper bounds, respectively, of $\gamma_R^P(G\overline{G})$ for a general graph G.

Theorem 2.14. *For any graph G,*

$$1 + \max\{\gamma_R^P(G), \gamma_R^P(\overline{G})\} \leq \gamma_R^P(G\overline{G}) \leq \rho,$$

where $\rho = \min\{\omega_G(f) + n - |V_2| : f = (V_0, V_1, V_2) \in PRD(G) \cup PRD(\overline{G})\}$.

Proof: WLOG assume that for some $f = (V_0, V_1, V_2)$ on G, $\rho = \omega_G(f) + n - |V_2|$. Extend f to $G\overline{G}$ by defining $f(\overline{v}) = 0$ for all $v \in V_2$ and $f(\overline{v}) = 1$ for all $v \in V(G) \setminus V_2$. Then the extension $f \in PRD(G\overline{G})$ and $\gamma_R^P(G\overline{G}) \leq \omega_G(f) + n - |V_2|$. Thus, $\gamma_R^P(G\overline{G}) \leq \rho$.

In view of Proposition 2.13(iv), we assume that neither G nor $\overline{G}$ is a complete graph. WLOG, assume that $\gamma_R^P(G) \geq \gamma_R^P(\overline{G})$. Let $f = (V_0, V_1, V_2)$ be a γ_R^P-function on $G\overline{G}$. If $V(\overline{G}) \subseteq V_0$, then $V_2 = V(G)$ so that $\gamma_R^P(G\overline{G}) = 2|V_2| = |V(G\overline{G})|$. Since $G\overline{G}$ is connected, $n = 1$ by Corollary 2.5 and Corollary 2.3(ii). This is contradictory to our assumption. Thus, $V(\overline{G}) \cap (V_1 \cup V_2) \neq \varnothing$. If $V_2 \cap V(\overline{G}) = \varnothing$, then $g = (V_0 \cap V(G), V_1 \cap V(G), V_2) \in PRD(G)$. Since $V(\overline{G}) \cap V_1 \neq \varnothing$,

$$\gamma_R^P(G\overline{G}) = \omega_{G\overline{G}}(f) \geq \omega_G(g) + 1 \geq \gamma_R^P(G) + 1.$$

Suppose that $V_2 \cap V(\overline{G}) \neq \varnothing$, and let $A = \{v \in V_0 : V_2 \cap N_{G\overline{G}}(v) = \{\overline{v}\}\}$. Define $g = (V_0^*, V_1^*, V_2^*)$ on G by

$$g(x) = \begin{cases} f(x), & \text{if } x \in V(G) \setminus A; \\ 1, & \text{if } x \in A. \end{cases}$$

Then $g \in PRD(G)$ with $V_0^* = (V_0 \setminus A) \cap V(G)$, $V_1^* = A \cup (V_1 \cap V(G))$ and $V_2^* = V_2 \cap V(G)$. Since $\{\overline{v} : v \in A\} \subseteq V_2 \cap V(\overline{G})$,

$$\gamma_R^P(G\overline{G}) = \omega_G(g) + \sum_{x \in V(\overline{G})} f(x) - |A| \geq \omega_G(g) + 1 \geq \gamma_R^P(G) + 1.$$

■

If $G = C_5$, then G and $\overline{G}$ are isomorphic and $G\overline{G}$ is isomorphic to the Petersen graph. Observe that $\gamma_R^P(G\overline{G}) = 7$, $\gamma_R^P(G) = \gamma_R^P(\overline{G}) = 4$ and $\rho = 8$ so that

$$1 + \max\{\gamma_R^P(G), \gamma_R^P(\overline{G})\} < \gamma_R^P(G\overline{G}) < \rho.$$

This shows that strict inequality can be attained at each side of the inequalities in Theorem 2.14.

2.4. On the edge corona of graphs

Given graphs G and H, we write H^{uv} to denote that copy of H that is being joined with the endvertices of the edge $uv \in E(G)$ in the edge corona $G \diamond H$. If $H = \{x\}$, then we write $V(H^{uv}) = \{x^{uv}\}$.

For an $f \in PRD(G)$, we write for each $a, b \in \{0, 1, 2\}$,

$$E_{ab}(f; G) = \{uv \in E(G) : (f(u) = a \wedge f(v) = b) \vee (f(u) = b \wedge f(v) = a)\},$$

where "$\wedge$" and "$\vee$" denote "and" and "or", respectively.

Theorem 2.15. *Let G be a nontrivial connected graph and H any graph of order n. Then*

$$\gamma_R^P(G \diamond H) \leq \alpha,$$

where

$$\alpha = \min_{g \in PRD(G)} \left(\omega_G(g) + |E_{11}(g; G)|\gamma_R^P(H) + n\left(|E_{01}(g; G)| + |E_{22}(g; G)| + E_{00}(g; G)|\right)\right),$$

and this upper bound is sharp.

Proof: Let $g \in PRD(G)$. If no confusion arises, we write $E_{ab} = E_{ab}(g; G)$. Let $h \in PRD(H)$. For each $ab \in E(G)$, we define a copy h^{ab} of h on H^{ab}. Define the function $f = (V_0, V_1, V_2)$ on $G \diamond H$ by

$$f(x) = \begin{cases} g(x), & \text{if } x \in V(G); \\ h^{uv}(x), & \text{if } x \in V(H^{uv}), \text{ where } uv \in E_{11}; \\ 0, & \text{if } x \in V(H^{uv}), \text{ where } uv \in E_{02} \cup E_{12}; \\ 1, & \text{if } x \in V(H^{uv}), \text{where } uv \in E_{01} \cup E_{00} \cup E_{22}. \end{cases}$$

We claim that $f \in PRD(G \diamond H)$. First, note that $f|_G = g = (V_0 \cap V(G), V_1 \cap V(G), V_2 \cap V(G))$. Let $x \in V_0$. Suppose that $x \in V(G)$. Then $N_{G\diamond H}(x) = N_G(x) \cup \big(\cup_{u\in N_G(x)} V(H^{ux})\big)$. Since $g \in PRD(G)$, $|V_2 \cap N_G(x)| = 1$, say $V_2 \cap N_G(x) = \{z\}$. Let $u \in N_G(x)$, and let $y \in V(H^{xu})$. If $u \in V_0 \cup V_1$, then $y \in V_1$. On the other hand, if $u \in V_2$, then $y \in V_0$. Thus, $V_2 \cap V(H^{ux}) = \varnothing$. Since u is arbitrary, $V_2 \cap \big(\cup_{u\in N_G(x)} V(H^{ux})\big) = \varnothing$ and so $V_2 \cap N_{G\diamond H}(x) = \{z\}$. Suppose that $x \in V(H^{uv})$ for some $uv \in E(G)$. Then $N_{G\diamond H}(x) = \{u, v\} \cup N_{H^{uv}}(x)$. Since $f(x) = 0$, $uv \notin E_{00} \cup E_{22} \cup E_{01}$. If $uv \in E_{11}$, then $h^{uv}(x) = 0$ and there exists exactly one $y \in V(H^{uv})$ such that $xy \in E(H^{uv})$ and $f(y) = h^{uv}(y) = 2$. In this case, $V_2 \cap N_{G\diamond H}(x) = V_2 \cap N_{H^{uv}}(x) = \{y\}$. Suppose that $uv \in E_{02} \cup E_{12}$. Since $V(H^{uv}) \subseteq V_0$, either $V_2 \cap N_{G\diamond H}(x) = \{u\}$ or $V_2 \cap N_{G\diamond H}(x) = \{v\}$. Accordingly, $f \in PRD(G \diamond H)$. Therefore,

$$\begin{aligned}\gamma_R^P(G \diamond H) &\le \omega_G(g) + |E_{11}|\omega_H(h) + \sum_{x\in\{V(H^{uv}):uv\in E_{00}\cup E_{01}\cup E_{22}\}} f(x)\\ &= \omega_G(g) + |E_{11}|\omega_H(h) + n\left(|E_{01}| + |E_{22}| + E_{00}|\right).\end{aligned}$$

Since h is arbitrary, the desired inequality holds.

Consider the graph $G \diamond P_3$ in Figure 2, where G is the caterpillar $ca(2, 0, 2)$ with the corresponding vertex labelling. The function g on $V(G)$ given by $g(x) = g(z) = 2$, $g(y) = 1$ and $g(x) = 0$ else is in $PRD(G)$. Since $E_{00} = E_{01} = E_{22} = E_{00} = \varnothing$, $\alpha \le \omega_G(g) = 5$ so that $\gamma_R^P(G \diamond P_3) \le 5$. Now, note that $\{x, z\}$ is the unique γ-set of $G \diamond P_3$. However, $\{x, z\}$

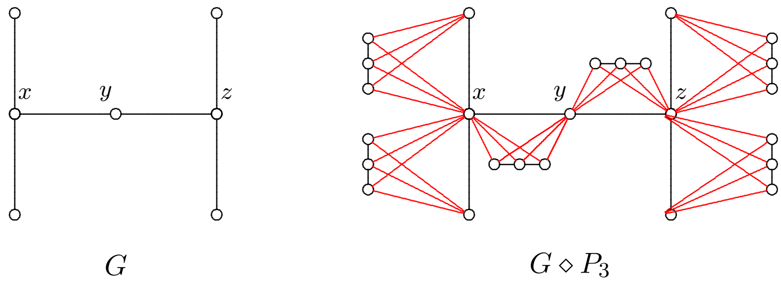

Figure 2: The edge corona $G \diamond P_3$ with $\gamma_R^P(G \diamond P_3) = 5$

does not form the $V_1 \cup V_2$ for any $f = (V_0, V_1, V_2) \in PRD(G \diamond P_3)$. Thus, $\gamma_R^P(G \diamond P_3) \ge 5$. ■

The value of α in Theorem 2.15 is not necessarily determined by a γ_R^P-function on G. Consider the two copies of the edge corona $P_5 \diamond C_4$ given in Figure 3 with the corresponding assignment of colours to the vertices. Here, we write $P_5 = \{x_1, x_2, x_3, x_4, x_5\}$. Observe that $f = (\{x_1, x_3, x_4\}, \varnothing, \{x_2, x_5\})$ is a γ_R^P-function on P_5 (see right-hand side figure), while $g = (\{x_1, x_5\}, \{x_3\}, \{x_2, x_4\}) \in PRD(P_5)$ but not a γ_R^P-function on P_5 (see left-hand side figure). Verify that $\gamma_R^P(P_5 \diamond C_4) = 5$ and is determined by the function g.

From Theorem 2.15 and as illustrated in the preceding example, the value of α in Theorem 2.15 is determined by the functions $g \in PRD(G)$ for which most of the sets

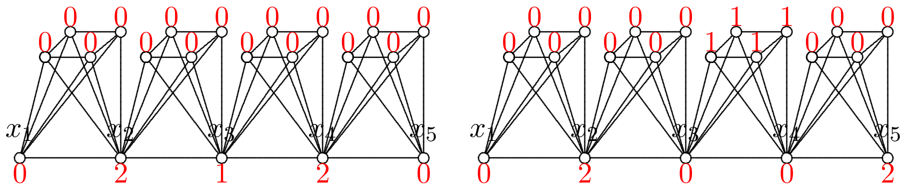

Figure 3: The edge corona $P_5 \diamond C_4$

$E_{00}(g;G)$, $E_{22}(g;G)$, $E_{11}(g;G)$ and $E_{01}(g;G)$ are empty. In view of such, the following observation can be easily verified.

Corollary 2.16. *Let H be any nontrivial graph of order m. then*

(i) For the path P_n on $n \geq 2$ vertices, $\gamma_R^P(P_n \diamond H) = 3\lfloor \frac{n-2}{2} \rfloor + 2$.

(ii) If $m \geq 3$, then for the cycle C_n on $n \geq 3$ vertices,

$$\gamma_R^P(C_n \diamond H) = \begin{cases} 3k, & \text{if } n = 2k; \\ 3k + 1 + \gamma_R^P(H), & \text{if } n = 2k+1. \end{cases}$$

(iii) If $m \geq 3$, then for $2 \leq n \leq k$, $\gamma_R^P(K_{n,k} \diamond H) = 2n + k$.

Theorem 2.17. *Let G be a nontrivial connected graph. Then*

$$\gamma_R^P(G \diamond K_1) = \min_{g \in PRD(G)} \left(\omega_G(G) + |E_{00}(g;G)| + |E_{01}(g;G)| + |E_{11}(g;G)| + |E_{22}(g;G)|\right).$$

Proof: Put

$$\alpha = \min\{\omega_G(G) + |E_{00}(g;G)| + |E_{01}(g;G)| + |E_{11}(g;G)| + |E_{22}(g;G)| : g \in PRD(G)\}.$$

By Theorem 2.15, $\gamma_R^P(G \diamond K_1) \leq \alpha$.

Let $f = (V_0, V_1, V_2)$ be a γ_R^P-function on $G \diamond K_1$. Suppose that the restriction $f|_G$ of f to G is not a perfect Roman dominating function on G. We will construct a γ_R^P-function g on $G \diamond K_1$ such that $\omega_{G\diamond K_1}(g) = \omega_{G\diamond K_1}(f)$ and its restriction $g|_G$ to G is a perfect Roman dominating function on G. There exists $u \in V_0 \cap V(G)$ such that $uv \notin E(G)$ for all $v \in V_2 \cap V(G)$. This means that there exists $v \in N_G(u)$ such that $V_2 \cap N_{G\diamond K_1}(u) = \{x^{uv}\}$.

Case 1: Suppose that $v \notin V_0$. Define $f^1 = (V_0^1, V_1^1, V_2^1)$ on $G \diamond K_1$ by $f^1(u) = f^1(x^{uv}) = 1$ and $f^1(x) = f(x)$ for all $x \in V(G \diamond K_1) \setminus \{u, x^{uv}\}$. Then $f^1 \in PRD(G \diamond K_1)$ with $\omega_{G\diamond K_1}(f^1) = \omega_{G\diamond K_1}(f)$.

Case 2: Suppose that $v \in V_0$. If $(N_G(v) \setminus \{u\}) \cap V_0 = \varnothing$, then take $f^1 = (V_0^1, V_1^1, V_2^1)$ on G given by $f^1(v) = 2$, $f^1(x^{uv}) = 0$ and $f^1(x) = f(x)$ for all $x \in V(G\diamond K_1)\setminus\{v, x^{uv}\}$. Then $f^1 \in PRD(G\diamond K_1)$ and $\omega_{G\diamond K_1}(f^1) = \omega_{G\diamond K_1}(f)$. Suppose that $B = (N_G(v) \setminus \{u\}) \cap V_0 \neq \varnothing$.

Necessarily, $x^{vw} \in V_1$ for each $w \in B$. In this case, take the function $f^1 = (V_0^1, V_1^1, V_2^1)$ on $G \diamond K_1$ given by

$$f^1(x) = \begin{cases} 2, & \text{if } x = v; \\ 0, & \text{if } x \in \{x^{uv}, x^{vw} : w \in B\}; \\ 1, & \text{if } x \in B; \\ f(x), & \text{if } x \in V(G \diamond K_1) \setminus (B \cup \{x^{vw} : w \in B\}). \end{cases}$$

Then $f^1 \in PRD(G \diamond K_1)$ with $V_0^1 = (V_0 \setminus \{v\}) \cup \{x^{uv}, x^{vw} : w \in B\}$, $V_1^1 = (V_1 \setminus \{x^{vw} : w \in B\}) \cup B$ and $V_2^1 = (V_2 \setminus \{x^{uv}\}) \cup \{v\}$. It is easy to verify that $f^1 \in PRD(G \diamond K_1)$ and $\omega_{G \diamond K_1}(f^1) = \omega_{G \diamond K_1}(f)$.

If $f^1|_G \notin PRD(G)$, then we follow the same process and obtain $f^2 \in PRD(G \diamond K_1)$ with $\omega_{G \diamond K_1}(f^2) = \omega_{G \diamond K_1}(f^1) = \omega_{G \diamond K_1}(f)$. If necessary, we do a finitely many repetitions of the process until we obtain a function $g = f^k \in PRD(G \diamond K_1)$ for which $\omega_{G \diamond K_1}(g) = \omega_{G \diamond K_1}(f)$ and $g|_G \in PRD(G)$. By the definition of α, $\gamma_R^P(G \diamond K_1) = \omega_{G \diamond K_1}(g) \geq \alpha$. ■

The value of $\gamma_R^P(G \diamond K_1)$ in Theorem 2.17 is determined by the functions $g \in PRD(G)$ for which the sets E_{22} and E_{11} are empty. With this observation, it can readily be verified that for $n \geq 1$ and $m \geq 3$,

$$\gamma_R^P(P_n \diamond K_1) = \lfloor \frac{n-1}{3} \rfloor + \gamma_R^P(P_n) \text{ and } \gamma_R^P(C_m \diamond K_1) = \lceil \frac{n}{3} \rceil + \gamma_R^P(C_m).$$

2.5. On the composition of graphs

Given $S \subseteq V(G[H])$, we write $S_G = \{x \in V(G) : (x, y) \in S \text{ for some } y \in V(H)\}$, which is called the *projection of* G on $G[H]$.

Proposition 2.18. *Let G and H be connected graphs, G noncomplete and H of order n with $\gamma(H) = 1$. Then*

$$\gamma_R^P(G[H]) \leq \alpha,$$

where $\alpha = \min\{(n-1)(|V_1| + |V_2 \cap N_G(V_2)|) + \omega_G(f) : f = (V_0, V_1, V_2) \in PRD(G)\}$.

Proof: Let $v \in V(H)$ for which $N_H[v] = V(H)$. Let $f = (V_0, V_1, V_2) \in PRD(G)$ such that $V_2 \neq \varnothing$. Define $g = (V_0^*, V_1^*, V_2^*)$ on $G[H]$ by

$$g((x, y)) = \begin{cases} 0, & \text{if } (x \in V_2 \setminus N_G(V_2) \ \wedge \ y \neq v) \ \vee \ (x \in V_0); \\ 1, & \text{if } (x \in V_2 \cap N_G(V_2) \ \wedge \ y \neq v) \ \vee (x \in V_1); \\ 2, & \text{if } x \in V_2 \text{ and } y = v. \end{cases}$$

with $V_0^* = ((V_2 \setminus N_G(V_2)) \times (V(H) \setminus \{v\})) \cup (V_0 \times V(H))$, $V_2^* = V_2 \times \{v\}$ and $V_1^* = (V_1 \cup V(H)) \cup ((V_2 \cap N_G(V_2)) \times (V(H) \setminus \{v\}))$. Let $(x, y) \in V_0^*$. If $x \in V_2$, then $x \notin N_G(V_2)$ so that $N_{G[H]}((x, y)) \cap V_2^* = \{(x, v)\}$. If $x \in V_0$, then there exists $u \in V_2$ such that $N_G(x) \cap V_2 = \{u\}$, which implies that $N_{G[H]}((x, y)) \cap V_2^* = \{(u, v)\}$. Thus, $g \in PRD(G[H])$. Therefore, $\gamma_R^P(G[H]) \leq |V_1^*| + 2|V_2^*| = (n-1)(|V_1| + |V_2 \cap N_G(V_2)|) + \omega_G(f)$. Since f is arbitrary, the desired inequality is established. ■

Proposition 2.19. *Let G be a nontrivial connected graph and $p \geq 2$. Then*

$$\gamma_R^P(G[K_p]) = \alpha,$$

where $\alpha = \min\{(n-1)\,(|V_1| + |V_2 \cap N_G(V_2)|) + \omega_G(f) : f = (V_0, V_1, V_2) \in PRD(G)\}$.

Proof: Let $f = (V_0, V_1, V_2)$ be a γ_P^R-function on $V(G[H])$. Then $V_2 \neq \varnothing$ and $V_0 \neq \varnothing$. First, we claim that $(V_0)_G \cap (V_1)_G = \varnothing$. Suppose not, and let $(x,y) \in V_1$ be such that $(x,z) \in V_0$ for some $z \neq y$. There exists unique $(u,v) \in V_2$ for which $(x,z)(u,v) \in E(G[K_p]$. If $u = x$, then since $y \neq v$, $(x,y)(u,v) \in E(G[K_p])$. Thus, whether $u = x$ or $x \neq u$, $(x,y)(u,v) \in E(G[K_p])$. By Proposition 2.1, there exists $(a,b) \in V_2 \setminus \{(u,v)\}$ such that $(x,y)(a,b) \in E(G[K_p])$. Using the same argument, whether $x = a$ or $x \neq b$, $(x,z)(a,b) \in E(G[K_p])$. This is a contradiction since $(x,z) \in V_0$.

Fix $v \in V(K_p)$. Define $A = \{(x,v) : x \in (V_0)_G \cap (V_2)_G\}$, $B = \{(x,y) \in V_2 : x \notin (V_0)_G\}$ and $C = \{(x,y) \in V_2 : x \in (V_0)_G\,, y \neq v\}$. Put

$$V_0^* = (V_0 \setminus A) \cup C, \;\; V_1^* = V_1, \;\; \text{and} \;\; V_2^* = A \cup B.$$

Then $\{V_0^*, V_1^*, V_2^*\}$ forms a partition of $V(G[K_p])$. Note here that, in particular, since $(V_0)_G \cap (V_1)_G = \varnothing$ and $V_1 \cap V_2 = \varnothing$. Now, let $(x,y) \in V_0^*$.

Case 1: Suppose that $(x,y) \in V_0 \setminus A$. There exists $(u,w) \in V_2$ such that $N_{G[K_p]}((x,y)) \cap V_2 = \{(u,w)\}$. If $u \notin (V_0)_G$, then $(u,w) \in B$ and $N_{G[K_p]}((x,y)) \cap V_2^* = \{(u,w)\}$. On the other hand, if $u \in (V_0)_G$, then $(u,v) \in A$ and $N_{G[K_p]}((x,y)) \cap V_2^* = \{(u,v)\}$.

Case 2: Suppose that $(x,y) \in C$ and let $z \in V(K_p) \setminus \{y\}$ for which $(x,z) \in V_0$. Since $(x,y)(x,z) \in E(G[K_p])$ and $(x,y) \in V_2$, $N_{G[K_p]}((x,z)) \cap V_2 = \{(x,y)\}$. This means that $(x,w) \notin V_2$ for all $w \in V(K_p) \setminus \{y\}$ and $(u,w) \notin V_2$ for all $u \in N_G(x)$ and for all $w \in V(K_p)$. Thus, $N_{G[K_p]}((x,y)) \cap V_2^* = N_{G[K_p]}((x,y)) \cap A = \{(x,v)\}$.

Accordingly, the function $g = (V_0^*, V_1^*, V_2^*) \in PRD(G[K_p])$. Since $V_1^* = V_1$ and $|V_2^*| \leq |V_2|$, $\omega_{G[K_p]}(f) \geq \omega_{G[K_p]}(g)$. Because f is a γ_R^P-function of $G[K_p]$, $\omega_{G[K_p]}(f) = \omega_{G[K_p]}(g)$ and g is a γ_R^P-function of $G[K_p]$.

Define the function $h = (V_0^h, V_1^h, V_2^h)$ on G by

$$h(x) = \begin{cases} 2, & \text{if } x \in (V_2^*)_G\,; \\ 1, & \text{if } x \in (V_1^*)_G \setminus (V_2^*)_G\,; \\ 0, & \text{else.} \end{cases}$$

Let $x \in V_0^h$. Then $(x,y) \in V_0^*$ for all $y \in V(K_p)$. Pick $y \in V(K_p)$. There exists a unique $(u,v) \in V_2^*$ for which $(x,y)(u,v) \in E(G[K_p])$. It follows that $u \in V_2^h$ and $ux \in E(G)$. Moreover, u is unique in this sense as (u,v) is for (x,y). Thus, $h \in PRD(G)$.

Finally, let $x, u \in V_2^h$ for which $xu \in E(G)$. Let $y, v \in V(K_p)$ such that $(x,y), (u,v) \in V_2^*$. Since g is a γ_R^P-function of $G[K_p]$, $(x,a), (u,b) \in V_1^*$ for all $a \in V(K_p) \setminus \{y\}$ and for all

$b \in V(K_p) \setminus \{v\}$. On the other hand, by the definition of h, for each $x \in V_1^h$, $(x, y) \in V_1^*$ for all $y \in V(K_p)$. Thus, $|V_1^*| \geq p|V_1^h| + (p-1)|V_2^h \cap N_G(V_2^h)|$. Therefore,

$$\begin{aligned} \gamma_R^P(G[K_p]) = \omega_{G[K_p]}(g) &= |V_1^*| + 2|V_2^*| \\ &\geq p|V_1^h| + (p-1)|V_2^h \cap N_G(V_2^h)| + 2|V_2^h| \\ &= (p-1)\left(|V_1^h| + |V_2^h \cap N_G(V_2^h)|\right) + \omega_G(h) \\ &\geq \alpha. \end{aligned}$$

The desired equality is completed by Proposition 2.18 ■

Equality in Proposition 2.18 is possible even if H is not complete. Consider the graph $G[P_3]$ in Figure 4, with G being the caterpillar graph $ca(0, 2, 0, 2, 0)$. Observe that $\alpha = 7$.

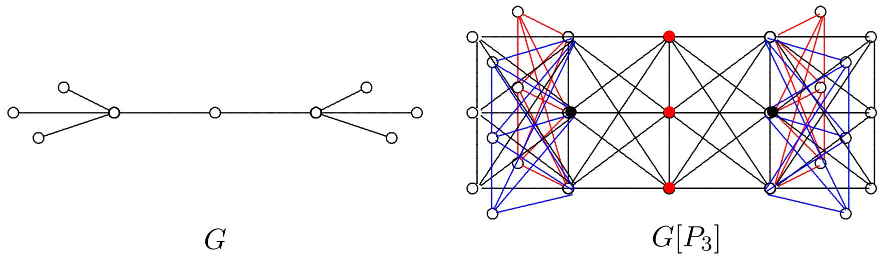

Figure 4: Graph G with $\gamma_R^P(G[P_3]) = 7$

On the other hand, $\gamma_R^P(G[P_3]) = 7$, which is determined by $(V_0, V_1, V_2) \in PRD(G[P_3])$, where V_1 and V_2 are the sets of all red and all black vertices, respectively, in $G[P_3]$ and $V_0 = V(G[P_3]) \setminus (V_1 \cup V_2)$.

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$\mu_{mn}S_p$-Open Sets in Bigeneralized Topological Spaces

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Abstract

In this paper, we introduce and characterize the notion of $\mu_{mn}S_p$-Open Sets, $\mu_{mn}S_p$-interior, and $\mu_{mn}S_p$-closure of a set in Bigeneralized Topological Spaces.

1 Introduction

In 2002, Császár introduced the concept of generalized topology [5]. Several counterparts of existing concepts in topology were defined including the μ-semiopen sets and μ-preopen sets.

Benjamin and Rara [4] introduced and characterizes the concepts of μS_p-open sets, μS_p-closed sets, μS_p-interior and μS_p-closure of a set in the generalized topological spaces. These concepts are generalized topology's counterpart of the S_p-open sets in [7].

Boonpok [3] introduced the concept of bigeneralized topological spaces. In this paper, we introduce and characterize the notions of $\mu_{mn}S_p$-Open Sets, $\mu_{mn}S_p$-interior, and $\mu_{mn}S_p$-closure of a set in Bigeneralized Topological Spaces.

Key words and phrases: $\mu_{mn}S_p$-Open sets, $\mu_{mn}S_p$-interior, and $\mu_{mn}S_p$-closure, Bigeneralized Topological Spaces
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2 Preliminaries

Let X be a nonempty set. A subset μ of $P(X)$ is said to be a generalized topology (briefly GT) on X if $\varnothing \in \mu$ and the arbitrary union of elements of μ belongs to μ. If μ is a GT on X, then the pair (X, μ) is called a generalized topological space (briefly GT-space), and the elements of μ are called μ-open sets. The complement of a μ-open set is called a μ-closed set.

Throughout this paper, the space (X, μ_1, μ_2) (or simply X) mean a bigeneralized topological space (BGT-space) with no separation axioms unless otherwise stated. Let A be a subset of a bigeneralized topological spaces. The closure and the interior of A with respect to μ_m are denoted by $c_{\mu_m}(A)$ and $i_{\mu_m}(A)$, respectively, with $m = 1, 2$.

In 2019, Fathima et. al [2] introduced the following definition:
Let (X, μ_1, μ_2) be a bigeneralized topological space. Let A be a subset of X. Then A is said to μ_{mn}-semiopen if $A \subseteq c_{\mu_m}(i_{\mu_n}(A))$, where $m, n = 1, 2$ and $m \neq n$. The complement of a μ_{mn}-semiopen set is called a μ_{mn}-semiclosed set.

Moreover, in 2020, Rani et. al [1] introduced the notion of a μ_{mn}-preopen set as follows:
Let (X, μ_1, μ_2) be a bigeneralized topological space. Let A be a subset of X. Then A is said to μ_{mn}-preopen if $A \subseteq i_{\mu_m}(c_{\mu_n}(A))$, where $m, n = 1, 2$ and $m \neq n$. The complement of a μ_{mn}-preopen set is called a μ_{mn}-preclosed set.

3 $\mu_{mn}S_p$-Open Sets in the Bigeneralized Topological spaces

In this section, we introduce the notion of $\mu_{mn}S_p$-Open Sets in the Bigeneralized Topological spaces.

Definition 3.1. A subset A of a bigeneralized topological space X is called $\mu_{mn}S_p$*-open* if A is μ_n-semiopen and for every $x \in A$, there exists a μ_m-preclosed set F_x such that $x \in F_x \subseteq A$. The complement of a $\mu_{mn}S_p$-open set is called a $\mu_{mn}S_p$*-closed* set.

Remark 3.2. *Let (X, μ_m, μ_n) be a bigeneralized topological space. Then A is a $\mu_{mn}S_p$-open set in X if and only if A is μ_n-semiopen and $A = \bigcup_{x \in A} F_x$, where F_x is a μ_m-preclosed set.*

Remark 3.3. *The concepts of $\mu_m S_p$-open set or $\mu_n S_p$-open set and the $\mu_{mn} S_p$-open sets are independent notions.*

Remark 3.4. *The $\mu_{12}S_p$-open sets need not be $\mu_{21}S_p$-open. To see this, let $X = \{a, b, c, d\}$ with generalized topologies $\mu_1 = \{\varnothing, \{a, c\}, \{d\}, \{a, c, d\}\}$ and $\mu_2 = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. Then $\{a, c, d\}$ is a $\mu_{21}S_p$-open set but not $\mu_{12}S_p$-open.*

Theorem 3.5. *Let (X, μ_m, μ_n) be a bigeneralized topological space. Then A is a $\mu_{mn}S_p$-closed set in X if and only if A is μ_n-semiclosed and for every $x \notin A$, there exists a μ_m-preopen set U_x such that $A \subseteq U_x$.*

Proof.
Let A be a $\mu_{mn}S_p$-closed set in X. Then $X\backslash A$ is $\mu_{mn}S_p$-open. By Definition 3.1, $X\backslash A$ is μ_n-semiopen and for every $x \in X\backslash A$, there exists a μ_m-preclosed set F_x such that $x \in F_x \subseteq X\backslash A$. Hence A is μ_n-semiclosed and for every $x \notin A$, there exists a μ_m-preopen set $X\backslash F_x$ such that $A \subseteq X\backslash F_x$. Take $U_x = X\backslash F_x$. Thus the necessity of the theorem follows. The sufficiency is proved similarly. This completes the proof. □

Definition 3.6. The union of all the $\mu_{mn}S_p$-open sets of a BGT-space X contained in $A \subseteq X$ is called the $\mu_{mn}S_p$*-interior of* A, denoted by $\mu_{mn}S_p i_{\mu_{mn}}(A)$.

Remark 3.7. For any subset A of a BGT-space X, $\mu_{mn}S_p i_{\mu_{mn}}(A) \subseteq A$.

Definition 3.8. The intersection of all the $\mu_{mn}S_p$-closed sets of X containing A is called the $\mu_{mn}S_p$*-closure* of A, denoted by $\mu_{mn}S_p c_{\mu_{mn}}(A)$.

Remark 3.9. For any subset A of a BGT-space X, $A \subseteq \mu_{mn}S_p c_{\mu_{mn}}(A)$.

4 $\mu_{mn}S_p$-Interior and $\mu_{mn}S_p$-Closure of a Set

In this section, we present some results involving $\mu_{mn}S_p$-interior and $\mu_{mn}S_p$-closure of a set in the BGT-space. First, consider the following remark:

Remark 4.1. Let (X, μ_m, μ_n) be a BGT-space and $A \subseteq X$. Then
(i) $\mu_{mn}S_p c_{\mu_{mn}}(A) = X\backslash \mu_{mn}S_p i_{\mu_{mn}}(X\backslash A)$;
(ii) $\mu_{mn}S_p i_{\mu_{mn}}(A) = X\backslash \mu_{mn}S_p c_{\mu_{mn}}(X\backslash A)$.
(iii) A is μ_{mn}-semiopen and μ_{mn}-preclosed if and only if $A = c_{\mu_m}(i_{\mu_n}(A))$.
(iv) If $A = c_{\mu_{mn}}(i_{\mu_{mn}}(A))$, then A is $\mu_{mn}S_p$-open.

The converse of Remark 4.1 (iv) need not be true. Consider the same BGT-space X in Remark 3.4. Observe that the set $A = \{a, b\}$ is $\mu_{mn}S_p$-open and $c_{\mu_{mn}}(i_{\mu_{mn}}(A)) = \{a, b, c\}$ which means $A \neq c_{\mu_{mn}}(i_{\mu_{mn}}(A))$.

Lemma 4.2. *Arbitrary union of μ_{mn}-semiopen sets is μ_{mn}-semiopen.*

Proof.
Let $\{M_i : i \in I\}$ be a collection of μ_{mn}-semiopen sets in a BGT-space X. Then $M_i \subseteq c_{\mu_m}(i_{\mu_n}(M_i))$ for all i. Thus

$$\begin{aligned} \bigcup_{i\in I} M_i &\subseteq \bigcup_{i\in I} c_{\mu_m}(i_{\mu_n}(M_i)) \\ &\subseteq c_{\mu_m}\left(\bigcup_{i\in I} i_{\mu_n}(M_i)\right) \\ &\subseteq c_{\mu_m}\left(i_{\mu_n}(\bigcup_{i\in I} M_i)\right). \end{aligned}$$

Therefore, $\bigcup_{i\in I} M_i$ is μ_{mn}-semiopen. □

Theorem 4.3. *The collection of all $\mu_{mn}S_p$-open sets in X forms a BGT on X.*

Proof.
Let $C = \{M_i : M_i \text{ is } \mu S_p\text{-open}, i \in I\}$. Clearly, $\varnothing$ is $\mu_{mn}S_p$-open. Since M_i is $\mu_{mn}S_p$-open for all $i \in I$, M_i is μ_{mn}-semiopen for all i. By Lemma 4.2, $\cup_{i\in I}M_i$ is μ_{mn}-semiopen. Let $x \in \cup_{i\in I}M_i$. Then $x \in M_i$ for some $i \in I$. Since M_i is $\mu_{mn}S_p$-open, there exists a μ_m-preclosed set F_i such that $x \in F_i \subseteq M_i$. This implies that $x \in F_i \subseteq \cup_{i\in I}M_i$. Therefore, $\cup_{i\in I}M_i$ is $\mu_{mn}S_p$-open. It follows that C forms a BGT on X. □

Corollary 4.4. *The intersection of all $\mu_{mn}S_p$-closed sets is $\mu_{mn}S_p$-closed.*

Proof.
Let F_i be $\mu_{mn}S_p$-closed sets for each $i \in I$. Then $X\backslash F_i$ is $\mu_{mn}S_p$-open for each i. By Theorem 4.3, $\cup_{i\in I}(X\backslash F_i) = X\backslash(\cap_{i\in I}F_i)$ is $\mu_{mn}S_p$-open. Therefore, $\cap_{i\in I}F_i$ is a $\mu_{mn}S_p$-closed set. □

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Separation Axioms via Generalized μS_p-Open Sets *

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Abstract

In this paper, we introduce and investigate some $g\mu S_p$-separation axioms in generalized topological spaces. Using the concepts of $g\mu S_p$-open sets, the study defines and characterizes $g\mu S_p$-R_0, $g\mu S_p$-R_1, $g\mu S_p$-T_0, $g\mu S_p$-T_1, $g\mu S_p$-T_2, $g\mu S_p$-*regular* and $g\mu S_p$-*normal* spaces.

Mathematics Subject Classification: 54A05

Keywords: $g\mu S_p$-R_0, $g\mu S_p$-R_1, $g\mu S_p$-T_0, $g\mu S_p$-T_1, $g\mu S_p$-T_2, $g\mu S_p$-*regular*, $g\mu S_p$-*normal*

1 Introduction

In [1], the concept of μS_p-sets and μS_p-functions was introduced. The objective of this paper is to introduce the concept of generalized μS_p-sets and investigate some of its properties. Furthermore, new separations axioms via the generalized μS_p-sets are defined and characterized. In particular, we will

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impose more conditions on a generalized topological space and determine the invariance properties of the resulting generalized topological space (GT-space).

For a subset A of a GT-space X, $\mu S_p c_\mu(A)$, $\mu S_p i_\mu(A)$, and $X \backslash A$ denote the μS_p-closure of A, μS_p-interior of A, and complement of A in X, respectively.

2 Preliminaries

Definition 2.1 [1] A subset A of a GT-space X is called *μS_p-open* if A is μ-semiopen and for every $x \in A$, there exists a μ-preclosed set F_x such that $x \in F_x \subseteq A$. The complement of a μS_p-open set is called a *μS_p-closed* set. If (X, μ_X) is a GT-space, the notation $\mu_X S_p$-open means a μS_p-open set in X with the generalized topology μ_X.

Remark 2.2 The collection of all μS_p-open sets in X forms a GT-space.

We will now define a larger set compared to a μS_p-open set. Some of its properties are established.

Definition 2.3 A subset A of a GT-space X is called a *generalized μS_p-closed set* (briefly $g\mu S_p$-closed) if $\mu S_p c_\mu(A) \subseteq U$ whenever U is μS_p-open with $A \subseteq U$. The complement of a $g\mu S_p$-closed set is called $g\mu S_p$-open.

Remark 2.4 Every μS_p-closed set is $g\mu S_p$-closed.

Remark 2.5 The collection of all $g\mu S_p$-open sets in X does not always form a GT on X.

Definition 2.6 The *$g\mu S_p$-closure* of a subset A of a GT-space X, denoted by $g\mu S_p c_\mu(A)$, is the intersection of all $g\mu S_p$-closed sets containing A.

Remark 2.7 The $g\mu S_p$-closure of a subset A of a GT-space X is not necessarily $g\mu S_p$-closed. Consider Let $X = \{a, b, c, d\}$ with the generalized topology $\mu = \{\varnothing, \{a, c\}, \{d\}, \{a, c, d\}\}$. Then μ-closed sets in X are $X, \{b, d\}$, $\{a, b, c\}$, and $\{b\}$. Thus $\{a, b, c\}$ and $\{b, c, d\}$ are $g\mu S_p$-closed sets but their intersection $\{b, c\}$ is not $g\mu S_p$-closed.

Definition 2.8 The union of all the μS_p-open sets of a GT-space X contained in $A \subseteq X$ is called the *μS_p-interior of* A, denoted by $\mu S_p i_\mu(A)$. The intersection of all the μS_p-closed sets of X containing A is called the *μS_p-closure* of A, denoted by $\mu S_p c_\mu(A)$.

Definition 2.9 A function $f : (X, \mu_X) \to (Y, \mu_Y)$ is called an *absolute $g\mu S_p$-continuous* if for every $g\mu_Y S_p$-open subset U of Y, $f^{-1}(U)$ is $g\mu_X S_p$-open in X.

Definition 2.10 A function $f : (X, \mu_X) \to (Y, \mu_Y)$ is called an *absolute $g\mu S_p$-open* if the image $f(A)$ is $g\mu_Y S_p$-open in Y for each $g\mu_X S_p$-open set A in X.

3 $g\mu S_p$-Separation Axioms

In this section, $g\mu S_p$-open sets are used to define separation axioms and some of their properties are established.

Definition 3.1 A GT-space X is called a

(i) $g\mu S_p$-R_0 if for each $g\mu S_p$-open set G and $x \in G$, $g\mu S_p c_\mu(\{x\}) \subseteq G$.

(ii) $g\mu S_p$-R_1 if for every $x, y \in X$ with $g\mu S_p c_\mu(\{x\}) \neq g\mu S_p c_\mu(\{y\})$, there exist disjoint $g\mu S_p$-open sets U and V such that $g\mu S_p c_\mu(\{x\}) \subseteq U$ and $g\mu S_p c_\mu(\{y\}) \subseteq V$.

(iii) $g\mu S_p$-T_0 *space* if for each pair of distinct points $x, y \in X$, there is either a $g\mu S_p$-open set containing x but not y or a $g\mu S_p$-open set containing y but not x.

(iv) $g\mu S_p$-T_1 *space* if for each pair of distinct points $x, y \in X$, there is a $g\mu S_p$-open set containing x but not y, and a $g\mu S_p$-open set containing y but not x.

(v) $g\mu S_p$-*Hausdorff* or $g\mu S_p$-T_2 *space* if for each pair of distinct points $x, y \in X$, there exists $g\mu S_p$-open sets U and V such that $x \in U, y \in V$ and $U \cap V = \varnothing$.

(vi) $g\mu S_p$-*regular space* if for each μS_p-closed subset $F \subseteq X$ and each point $x \notin F$, there exist $g\mu S_p$-open sets U and V such that $x \in U$, $F \subseteq V$ and $U \cap V = \varnothing$.

(vii) $g\mu S_p$-T_3 *space* if it is both $g\mu S_p$-T_1 and $g\mu S_p$-regular space.

(viii) $g\mu S_p$-*normal* if for each pair of disjoint μS_p-closed subsets F_1 and F_2, there exist $g\mu S_p$-open sets U and V such that $F_1 \subseteq U$, $F_2 \subseteq V$ and $U \cap V = \varnothing$.

(ix) $g\mu S_p$-T_4 *space* if it is both $g\mu S_p$-T_1 and $g\mu S_p$-normal space.

The relationship of $g\mu S_p$-R_0 and $g\mu S_p$-R_1 spaces is given in the next theorem.

Theorem 3.2 *Every $g\mu S_p$-R_1 space X is $g\mu S_p$-R_0.*

Proof: Suppose that X is a $g\mu S_p$-R_1 space. Let U be a $g\mu S_p$-open set in X and $x \in U$. Suppose that $g\mu S_p c_\mu(\{x\}) \nsubseteq U$. Then there exists $y \in g\mu S_p c_\mu(\{x\})$ such that $y \notin U$. Since $x \notin X\backslash U$ and $X\backslash U$ is $g\mu S_p$-closed containing y, $x \notin g\mu S_p c_\mu(\{y\})$ and so $g\mu S_p c_\mu(\{x\}) \neq g\mu S_p c_\mu(\{y\})$. Since X is $g\mu S_p$-R_1, there exists a $g\mu S_p$-open set V such that $g\mu S_p c_\mu(\{y\}) \subseteq V$ and $x \notin V$. Thus, $x \in X\backslash V$, $X\backslash V$ is $g\mu S_p$-closed and $y \notin X\backslash V$. This means that $y \notin g\mu S_p c_\mu(\{x\})$. This is a contradiction to $y \in g\mu S_p c_\mu(\{x\})$. Thus, $g\mu S_p c_\mu(\{x\}) \subseteq U$. Therefore, X is $g\mu S_p$-R_0. □

Theorem 3.3 *The following statements are equivalent for a GT-space X:*
(i) *X is a $g\mu S_p$-R_0 space.*
(ii) *$x \in g\mu S_p c_\mu(\{y\})$ if and only if $y \in g\mu S_p c_\mu(\{x\})$ for any two points x and y in X.*

Proof: (i) ⇒ (ii): Let $x \in g\mu S_p c_\mu(\{y\})$ and U be any $g\mu S_p$-open set such that $y \in U$. Since $g\mu S_p c_\mu(\{y\}) \subseteq U$, $x \in U$. Thus, $y \in g\mu S_p c_\mu(\{x\})$. Similarly, if $y \in g\mu S_p c_\mu(\{x\})$, then $x \in g\mu S_p c_\mu(\{y\})$.
(ii) ⇒ (i): Let $y \in g\mu S_p c_\mu(\{x\})$. By assumption, $x \in g\mu S_p c_\mu(\{y\})$. This implies that $y \in V$. Therefore, $g\mu S_p c_\mu(\{x\}) \subseteq V$. This proves that X is a $g\mu S_p$-R_0 space. □

We will introduce the concept of $g\mu S_p$-kernel of a set and use it to characterize the notions of $g\mu S_p$-R_0 and $g\mu S_p$-R_1.

Definition 3.4 If X is a GT-space and $A \subseteq X$, then the *$g\mu S_p$-kernel of A*, denoted by $g\mu S_p Ker(A)$, is defined to be the set

$$g\mu S_p Ker(A) = \cap\{U \subseteq X : U \text{ is } g\mu S_p\text{-open and } A \subseteq U\}.$$

The next result follows immediately from Definition 3.4.

Lemma 3.5 *If X is a GT-space and $x, y \in X$, then $y \in g\mu S_p Ker(\{x\})$ if and only if $x \in g\mu S_p c_\mu(\{y\})$.*

In Remark 2.5, the union of any collection of $g\mu S_p$-open sets need not be $g\mu S_p$-open. In the following definition, a property is defined so that the union of any collection of $g\mu S_p$-open sets is also $g\mu S_p$-open.

Definition 3.6 We say that the family of all $g\mu S_p$-open sets in a GT-space X has the property ϑ if the union of any collection of $g\mu S_p$-open sets in X is also $g\mu S_p$-open. Let $g\mu S_pO(X,\mu)$ denote the collection of $g\mu S_p$-open sets in a GT-space (X,μ).

Theorem 3.7 *Let X be a GT-space and $A \subseteq X$ such that $g\mu S_pO(X)$ has the property ϑ. Then $g\mu S_pKer(A) = \{x \in X : g\mu S_pc_\mu(\{x\}) \cap A \neq \varnothing\}$.*

Proof: Let $x \in g\mu S_pKer(A)$ and suppose on the contrary that $g\mu S_pc_\mu(\{x\}) \cap A = \varnothing$. Then $A \subseteq X \backslash g\mu S_pc_\mu(\{x\})$ and $X \backslash g\mu S_pc_\mu(\{x\})$ is a $g\mu S_p$-open set by assumption. Now, since $x \notin X \backslash g\mu S_pc_\mu(\{x\})$ and $x \in g\mu S_pKer(A)$, we obtain a contradiction.

On the other hand, let $x \in X$ such that $g\mu S_pc_\mu(\{x\}) \cap A \neq \varnothing$. Suppose that $x \notin g\mu S_pKer(A)$. Then there exists a $g\mu S_p$-open set U such that $A \subseteq U$ and $x \notin U$. This implies that for each $a \in A$, $a \notin g\mu S_pc_\mu(\{x\})$. Thus, $g\mu S_pc_\mu(\{x\}) \cap A = \varnothing$, contrary to our assumption. □

Theorem 3.8 *Let x, y be any two points in a GT-space X. If $g\mu S_pc_\mu(\{x\}) \neq g\mu S_pc_\mu(\{y\})$, then $g\mu S_pKer(\{x\}) \neq g\mu S_pKer(\{y\})$. Moreover, if $g\mu S_pO(X)$ has the property ϑ, then the converse holds.*

Proof: Suppose that $g\mu S_pc_\mu(\{x\}) \neq g\mu S_pc_\mu(\{y\})$. Assume that $g\mu S_pc_\mu(\{x\}) \nsubseteq g\mu S_pc_\mu(\{y\})$. Then there exists a point $z \in g\mu S_pc_\mu(\{x\})$ and $z \notin g\mu S_pc_\mu(\{y\})$. Thus, there exists a $g\mu S_p$-open set U containing z (and hence containing x) not y. Hence, $y \notin g\mu S_pKer(\{x\})$. Since $x \in g\mu S_pKer(\{x\})$, $x \notin g\mu S_pc_\mu(\{y\})$. By Lemma 3.5, $y \notin g\mu S_pKer(\{x\})$. Therefore, $g\mu S_pKer(\{x\}) \neq g\mu S_pKer(\{y\})$.

Conversely, suppose that $g\mu S_pKer(\{x\}) \neq g\mu S_pKer(\{y\})$. Then there exists $z \in X$ such that $z \in g\mu S_pKer(\{x\})$ but $z \notin g\mu S_pKer(\{y\})$. Since $z \in g\mu S_pKer(\{x\})$, by Theorem 3.7, $\{x\} \cap g\mu S_pc_\mu(\{z\}) \neq \varnothing$. Hence, $x \in g\mu S_pc_\mu(\{z\})$. Since $z \notin g\mu S_pKer(\{y\})$, $\{y\} \cap g\mu S_pc_\mu(\{z\}) = \varnothing$. Because $x \in g\mu S_pc_\mu(\{z\})$ and $g\mu S_pc_\mu(\{z\})$ is $g\mu S_p$-closed by assumption, $g\mu S_pc_\mu(\{x\}) \subseteq g\mu S_pc_\mu(\{z\})$ and $\{y\} \cap g\mu S_pc_\mu(\{z\}) = \varnothing$. Therefore, $g\mu S_pc_\mu(\{x\}) \neq g\mu S_pc_\mu(\{y\})$. This completes the proof. □

A characterization of $g\mu S_p$-R_0 space is given in the next result.

Theorem 3.9 *Let X be a GT-space. If $g\mu S_pO(X)$ has the property ϑ, then the following statements are equivalent:*

(i) *X is a $g\mu S_p$-R_0 space.*

(ii) *For any $x \in X$, $g\mu S_pc_\mu(\{x\}) \subseteq g\mu S_pKer(\{x\})$.*

(iii) *For any $g\mu S_p$-closed set F and a point $x \notin F$, there exists a $g\mu S_p$-open*

set U such that $x \notin U$ and $F \subseteq U$.
(iv) *If F is a $g\mu S_p$-closed set, then*

$$F = \cap\{U \subseteq X : U \text{ is } g\mu S_p\text{-open and } F \subseteq U\}.$$

(v) *If F is a $g\mu S_p$-closed set such that $x \notin F$, then $g\mu S_p c_\mu(\{x\}) \cap F = \varnothing$.*

Proof:
(i) $\Rightarrow$ (ii): If $y \in g\mu S_p c_\mu(\{x\})$, then $x \in g\mu S_p c_\mu(\{y\})$ by Theorem 3.3. By Lemma 3.5, $y \in g\mu S_p Ker(\{x\})$. Thus, (ii) holds.
(ii) $\Rightarrow$ (iii): Suppose that F is a $g\mu S_p$-closed set and $x \in X$ such that $x \notin F$. Then for $y \in F$, $g\mu S_p c_\mu(\{y\}) \subseteq F$. Thus, $x \notin g\mu S_p c_\mu(\{y\})$ so that $y \notin g\mu S_p c_\mu(\{x\})$. Hence, there exists a $g\mu S_p$-open set O with $y \in O$ but $x \notin O$ for every $y \in F$. Let $U = \cup\{O : O$ is $g\mu S_p$-open such that $y \in O, x \notin O\}$. By assumption, U is $g\mu S_p$-open such that $x \notin U$ and $F \subseteq U$.
(iii) $\Rightarrow$ (iv): Let F be any $g\mu S_p$-closed set and $W = \cap\{U \in g\mu S_p O(X) : F \subseteq U\}$. Then $F \subseteq W$. Let $x \notin F$. Then by hypothesis, there exists $U \in g\mu S_p O(X)$ such that $x \notin U$ and $F \subseteq U$. Hence, $x \notin W$. It follows that $W \subseteq F$.
(iv) $\Rightarrow$(v): Let F be a $g\mu S_p$-closed set with $x \notin F$. Then by (iv), $x \notin \cap\{U \in g\mu S_p O(X) : F \subseteq U\}$. Thus, there exists a $g\mu S_p$-open set H such that $x \notin H$ and $F \subseteq H$. Then $x \in X\backslash H = M \subseteq X\backslash F$ so that $g\mu S_p c_\mu(\{x\}) \subseteq M \subseteq X\backslash F$. Hence, $g\mu S_p c_\mu(\{x\}) \cap F = \varnothing$.
(v) $\Rightarrow$ (i): Let U be a $g\mu S_p$-open set such that $x \in U$. Then $X\backslash U$ is a $g\mu S_p$-closed set and $x \notin X\backslash U$. Hence, $g\mu S_p c_\mu(\{x\}) \cap X\backslash U = \varnothing$. Therefore, $g\mu S_p c_\mu(\{x\}) \subseteq U$. Consequently, X is a $g\mu S_p$-R_0 space. □

Theorem 3.10 *Suppose that $g\mu S_p O(X)$ has the property ϑ. A GT-space X is $g\mu S_p$-T_0 if and only if the $g\mu S_p$-closure of distinct points are distinct.*

Proof: Let X be a $g\mu S_p$-T_0 space. Suppose that $x, y \in X$ with $x \neq y$. Then there exists a $g\mu S_p$-open set U that contains x but not y. Then $X\backslash U$ is $g\mu S_p$-closed in X which contains y but not x. Since $\{y\} \subseteq X\backslash U$, $g\mu S_p c_\mu(\{y\}) \subseteq X\backslash U$ and since $x \notin X\backslash U$, $x \notin g\mu S_p c_\mu(\{y\})$. Hence, $g\mu S_p c_\mu(\{x\}) \neq g\mu S_p c_\mu(\{y\})$.

Conversely, let $x, y \in X$ such that $x \neq y$. By assumption, $g\mu S_p c_\mu(\{x\}) \neq g\mu S_p c_\mu(\{y\})$. Then there exists at least one point d of X such that $d \in g\mu S_p c_\mu(\{x\})$ and $d \notin g\mu S_p c_\mu(\{y\})$. If $x \in g\mu S_p c_\mu(\{y\})$, then $\{x\} \subseteq g\mu S_p c_\mu(\{y\})$. Hence, $g\mu S_p c_\mu\{x\} \subseteq g\mu S_p c_\mu(\{y\})$. This is a contradiction since $d \notin g\mu S_p c_\mu(\{y\})$ but $d \in g\mu S_p c_\mu\{x\}$. Thus, $x \notin g\mu S_p c_\mu(\{y\})$. Hence, $X\backslash g\mu S_p c_\mu(\{y\})$ is a $g\mu S_p$-open set containing x but not y. Therefore, X is $g\mu S_p$-T_0 space. □

Theorem 3.11 *Let X be a GT-space and $x \in X$ such that $g\mu S_p O(X)$ has the property ϑ. If $\{x\}$ is a $g\mu S_p$-open set for every $x \in X$, then X is a $g\mu S_p$-T_1 space.*

Proof: Suppose that for each $x \in X$, $\{x\}$ is a $g\mu S_p$-open set. If $|X| = 1$, then the result follows. Let $|X| > 1$ and let x, y be distinct points of X. Then $\{y\}$ is a $g\mu S_p$-open set and $X\backslash\{y\} = \cup_{x \in X\backslash\{y\}}\{x\}$ is a $g\mu S_p$-open set containing x but not y. Therefore, X is a $g\mu S_p$-T_1 space. □

Theorem 3.12 *Let X be a GT-space. If every singleton of X is $g\mu S_p$-closed, then X is $g\mu S_p$-T_1.*

Proof: Let $\{x\}$ be $g\mu S_p$-closed for every $x \in X$. Suppose that $x, y \in X$ are distinct. Then $X\backslash\{x\}$ is $g\mu S_p$-open containing y but not x. Also, $X\backslash\{y\}$ is $g\mu S_p$-open containing x but not y. Therefore, X is $g\mu S_p$-T_1. □

Remark 3.13 The converse of Theorem 3.12 is not necessarily true.

The next result provides additional condition so that the converse of Theorem 3.12 holds.

Theorem 3.14 *Let X be a GT-space such that $g\mu S_pO(X)$ has the property ϑ. If X is $g\mu S_p$-T_1, then every singleton subset of X is $g\mu S_p$-closed.*

Proof: Suppose that X is a $g\mu S_p$-T_1 space. Let $x \in X$ and $y \in X\backslash\{x\}$. Then $x \neq y$. Since X is $g\mu S_p$-T_1, there exists a $g\mu S_p$-open set U_y such that $y \in U_y$ and $x \notin U_y$. Thus, for each $y \in X\backslash\{x\}$, there exists a $g\mu S_p$-open set U_y such that $y \in U_y \subseteq X\backslash\{x\}$. It follows that $\cup\{\{y\}|y \neq x\} \subseteq \cup\{U_y|y \neq x\} \subseteq X\backslash\{x\}$. Hence, $X\backslash\{x\} = \cup\{U_y|y \neq x\}$. Since $g\mu S_pO(X)$ has the property ϑ, $X\backslash\{x\}$ is $g\mu S_p$-open. Therefore, $\{x\}$ is $g\mu S_p$-closed. □

The next corollary follows fro Theorem 3.12 and Theorem 3.14.

Corollary 3.15 *Let X be a GT-space such that $g\mu S_pO(X)$ has the property ϑ. Then X is $g\mu S_p$-T_1 if and only if every singleton subset of X is $g\mu S_p$-closed.*

It is well known that a topological space X is a T_1-space if and only if each finite subset of X is closed. Can "singleton" in Theorem 3.15 be replaced by "finite subset"? Remark 3.13 shows that the answer to this question is negative.

We will now characterize a $g\mu S_p$-T_1 space.

Theorem 3.16 *Let X be a GT-space such that $g\mu S_pO(X)$ has the property ϑ. The following statements are equivalent:*
(i) *X is a $g\mu S_p$-T_1 space.*
(ii) *Each subset of X is the intersection of all $g\mu S_p$-open sets containing it.*
(iii) *The intersection of all $g\mu S_p$-open sets containing the point $x \in X$ is $\{x\}$.*

Proof:
(i) $\Rightarrow$ (ii): Let X be a $g\mu S_p$-T_1 space and $A \subseteq X$. If $y \notin A$, there exists a set $X \backslash \{y\}$ such that $A \subseteq X \backslash \{y\}$. By Theorem 3.14, $X \backslash \{y\}$ is $g\mu S_p$-open for every y. Thus, $A = \cap\{X \backslash \{y\} : y \in X \backslash A\}$.
(ii) $\Rightarrow$ (iii): Let $x \in X$. Then $\{x\} \subseteq X$. By assumption,

$$\cap\{U \subseteq X : U \text{ is } g\mu S_p\text{-open with } x \in U\} = \{x\}.$$

(iii) $\Rightarrow$ (i): Suppose $x, y \in X$ such that $x \neq y$. Let

$$U_x = \cap\{U \subseteq X : U \text{ is } g\mu S_p\text{-open with } x \in U\}$$

and

$$U_y = \cap\{V \subseteq X : V \text{ is } g\mu S_p\text{-open with } y \in V\}.$$

By (iii), $U_x = \{x\}$ and $U_y = \{y\}$. Thus there exist $g\mu S_p$-open sets U_x and U_y with $x \in U_x, y \notin U_x, y \in U_y$, and $x \notin U_y$. Hence, X is a $g\mu S_p$-T_1 space. □

Remark 3.17 Every $g\mu S_p$-T_1 space is $g\mu S_p$-T_0, but the converse is not true. The next theorem provides a condition for a $g\mu S_p$-T_0 space to be $g\mu S_p$-T_1.

Theorem 3.18 *If a GT-space X is both $g\mu S_p$-T_0 and $g\mu S_p$-R_0, then X is a $g\mu S_p$-T_1 space.*

Proof: Let $x, y \in X$ be any pair of distinct points. Since X is a $g\mu S_p$-T_0 space, there exists a $g\mu S_p$-open set U such that $x \in U$ and $y \notin U$ or there exists a $g\mu S_p$-open set V such that $y \in V$ and $x \notin V$. Since X is $g\mu S_p$-R_0, $g\mu S_p c_\mu(\{x\}) \subseteq U$ and $y \notin g\mu S_p c_\mu(\{x\})$. Hence, $y \in W = X \backslash g\mu S_p c_\mu(\{x\})$. Since $g\mu S_p c_\mu(\{x\})$ is $g\mu S_p$-closed, W is $g\mu S_p$-open. Therefore, there exist $g\mu S_p$-open sets U and W containing x and y respectively such that $y \notin U$ and $x \notin W$. Therefore, X is a $g\mu S_p$-T_1 space. □

Remark 3.19 Every $g\mu S_p$-T_2 space is $g\mu S_p$-T_1, but the converse is not true. The next theorem characterizes a $g\mu S_p$-T_2 space.

Theorem 3.20 *Let X be a GT-space. The following statements are equivalent:*
(i) *X is a $g\mu S_p$-T_2 space.*
(ii) *For a given $x_0 \in X$ and for any $x \in X$ such that $x \neq x_0$, there exists a $g\mu S_p$-open set U in X with $x_o \in U$ and $x \notin g\mu S_p c_\mu(U)$.*
(iii) *For each $x \in X$,*

$$\cap\{g\mu S_p c_\mu(U) : U \text{ is } g\mu S_p \text{ -open in } X \text{ with } x \in X\} = \{x\}.$$

Proof:
(i) $\Rightarrow$ (ii): Let $x_0 \in X$ and consider $x \in X$ such that $x \neq x_0$. Since X is a $g\mu S_p$-T_2 space, there exist disjoint $g\mu S_p$-open sets U and V such that $x_0 \in U$ and $x \in V$. Then $X \backslash V$ is $g\mu S_p$-closed and $g\mu S_p c_\mu(U) \subseteq X \backslash V$. Since $x \notin X \backslash V$, $x \notin g\mu S_p c_\mu(U)$.
(ii) $\Rightarrow$ (iii): Let $x \in X$. Then for each $y \in X$ such that $y \neq x$, there exists a $g\mu S_p$-open set U such that $x \in U$ and $y \notin g\mu S_p c_\mu(U)$. Thus,

$$\cap\{g\mu S_p c_\mu(U) : U \text{ is } g\mu S_p \text{ -open in } X \text{ with } x \in X\} = \{x\}.$$

(iii) $\Rightarrow$ (i): Suppose that $x, y \in X$ such that $x \neq y$. Let

$$W_x = \cap\{g\mu S_p c_\mu(U) : U \text{ is } g\mu S_p\text{-open in } X \text{ with } x \in U\}.$$

By (iii), $W_x = \{x\}$. Hence, $y \notin W_x$. This implies that there exists $g\mu S_p$-open set U with $x \in U$ and $y \notin g\mu S_p c_\mu(U)$. Let $V = X \backslash g\mu S_p c_\mu(U)$. Then $U \cap V = \varnothing$. Since $g\mu S_p c_\mu(U)$ is a $g\mu S_p$-closed set, V is a $g\mu S_p$-open and $y \in V$. Therefore, X is a $g\mu S_p$-T_2 space. □

The next theorem establishes links between $g\mu S_p$-T_2 and $g\mu S_p$-R_1 spaces.

Theorem 3.21 *Let X be a $g\mu S_p$-T_0 space such that $g\mu S_p O(X)$ has the property ϑ. Then X is $g\mu S_p$-T_2 if and only if $g\mu S_p$-R_1.*

Proof: Let $x, y \in X$ such that $x \neq y$. By Theorem 3.10, $g\mu S_p c_\mu(\{x\}) \neq g\mu S_p c_\mu(\{y\})$. Since X is $g\mu S_p$-T_2, there exist disjoint $g\mu S_p$-open sets U and V such that $x \in U$ and $y \in V$. By Remark 3.19 and Theorem 3.15, $g\mu S_p c_\mu(\{x\}) = \{x\} \subseteq U$ and $g\mu S_p c_\mu(\{y\}) = \{y\} \subseteq V$. Hence, X is $g\mu S_p$-R_1.

Conversely, suppose that $x, y \in X$ with $x \neq y$. Then by Theorem 3.10, $g\mu S_p c_\mu(\{x\}) \neq g\mu S_p c_\mu(\{y\})$. Since X is $g\mu S_p$-R_1, there are disjoint $g\mu S_p$-open sets U and V such that $x \in g\mu S_p c_\mu(\{x\}) \subseteq U$ and $y \in g\mu S_p c_\mu(\{y\}) \subseteq V$. Therefore, X is $g\mu S_p$-T_2. □

The following theorem is a characterization of a $g\mu S_p$-R_1 space.

Theorem 3.22 *Let X be a $g\mu S_p$-R_1 space. Then for any $x, y \in X$ such that $g\mu S_p c_\mu(\{x\}) \neq g\mu S_p c_\mu(\{y\})$, there exist $g\mu S_p$-closed sets M and N such that $x \in M, y \notin M, y \in N, x \notin N$ and $X = M \cup N$. If in addition, $g\mu S_p O(X)$ has the property ϑ, then the converse holds.*

Proof: Suppose that $x, y \in X$ such that $g\mu S_p c_\mu(\{x\}) \neq g\mu S_p c_\mu(\{y\})$. Since X is a $g\mu S_p$-R_1 space, there exist disjoint $g\mu S_p$-open sets U and V such that $x \in g\mu S_p c_\mu(\{x\}) \subseteq U$ and $y \in g\mu S_p c_\mu(\{y\}) \subseteq V$. Let $M = X \backslash U$ and

$N = X\backslash V$. Then M and N are $g\mu S_p$-closed sets such that $x \notin M, y \in M$, $x \in N, y \notin N$ and $X = M \cup N$.

Conversely, let $x, y \in X$ be distinct such that $g\mu S_p c_\mu(\{x\}) \neq g\mu S_p c_\mu(\{y\})$. By assumption, there exist $g\mu S_p$-closed sets M and N such that $x \in M$, $y \notin M, y \in N, x \notin N$ and $X = M \cup N$. Let $U = X\backslash N$ and $V = X\backslash M$. Then U and V are $g\mu S_p$-open sets such that $x \in U, y \in V$ and $U \cap V = \varnothing$. Therefore, X is a $g\mu S_p$-R_1 by Theorem 3.21. □

4 $g\mu S_p$-Regular and $g\mu S_p$-Normal Spaces

In this section, we investigate the concepts of $g\mu S_p$-regular and $g\mu S_p$-normal spaces and establish some of their characterization.

Lemma 4.1 *Let (X, μ) be a GT-space and U, V are $g\mu S_p$-open sets in X. Then $U \cap V = \varnothing$ if and only if $g\mu S_p c_\mu(U) \cap V = \varnothing$.*

Proof: Suppose that $U \cap V = \varnothing$. Then $U \subseteq X\backslash V$ and $X\backslash V$ is $g\mu S_p$-closed. Thus, $g\mu S_p c_\mu(U) \subseteq g\mu S_p c_\mu(X\backslash V) = X\backslash V$. This means that $g\mu S_p c_\mu(U) \cap V = \varnothing$. The converse is clear. □

Theorem 4.2 *Let X be a GT-space. If X is a $g\mu S_p$-T_4 space, then X is $g\mu S_p$-T_3.*

Proof: Suppose that X is a $g\mu S_p$-T_4 space. Let $x \in X$ and F be a $g\mu S_p$-closed set such that $x \notin F$. Since X is $g\mu S_p$-T_1, $\{x\}$ is $g\mu S_p$-closed by Theorem 3.15. By normality of X, there exist $g\mu S_p$-open sets U and V such that $x \in U$, $F \subseteq V$, and $U \cap V = \varnothing$. Therefore, X is a $g\mu S_p$-T_3 space. □

Theorem 4.3 *The following statements are equivalent for a GT-space X where $g\mu S_p O(X)$ possesses the property ϑ.*

(i) *X is a $g\mu S_p$-regular space.*

(ii) *For every $x \in X$ and for every μS_p-closed set F such that $x \notin F$, there exist $g\mu S_p$-open sets U_x and V_F such that $x \in U_x$, $F \subseteq V_F$ and $g\mu S_p c_\mu(U_x) \cap V_F = \varnothing$.*

(iii) *For every $x \in X$ and for every μS_p-closed set F such that $x \notin F$, there exist $g\mu S_p$-open sets U_x such that $g\mu S_p c_\mu(U_x) \cap F = \varnothing$.*

(iv) *For every $x \in X$ and for every μS_p-open set U_x containing x, there exists a $g\mu S_p$-open set V_x containing x such that $x \in V_x \subseteq g\mu S_p c_\mu(V_x) \subseteq U_x$.*

v.) *For every $g\mu S_p$-closed set F of X,*

$$F = \cap\{g\mu S_p c_\mu(U) : F \subseteq U \textit{ and } U \textit{ is } g\mu S_p\textit{-open}\}.$$

Proof: (i) $\Rightarrow$ (ii): This follows from Lemma 4.1.
(ii) $\Rightarrow$ (iii): This is straightforward.
(iii) $\Rightarrow$ (iv): Let $x \in X$ and suppose that U_x is a μS_p-open set with $x \in U_x$. Then $X \backslash U_x$ is a μS_p-closed set and $x \notin X \backslash U_x$. Thus, there exists a $g\mu S_p$-open set V_x with $x \in V_x$ such that $g\mu S_p c_\mu(V_x) \cap (X \backslash U_x) = \varnothing$. It follows that $g\mu S_p c_\mu(V_x) \subseteq U_x$. Hence, $x \in V_x \subseteq g\mu S_p c_\mu(V_x) \subseteq U_x$.
(iv) $\Rightarrow$ (v): Let F be a $g\mu S_p$-closed subset of X. Then

$$F \subseteq \cap\{g\mu S_p c_\mu(U) : F \subseteq U \text{ and } U \text{ is } g\mu S_p\text{-open}\}.$$

Suppose that $x \notin F$. Then $x \in X \backslash F$ and $X \backslash F$ is a $g\mu S_p$-open set. By (iv), there exists a $g\mu S_p$-open set U_x with $x \in U_x$ such that

$$x \in U_x \subseteq g\mu S_p c_\mu(U_x) \subseteq X \backslash F.$$

Let $V_F = X \backslash g\mu S_p c_\mu(U_x)$. Since $g\mu S_p O(X)$ has the property ϑ, $g\mu S_p c_\mu(U_x)$ is a $g\mu S_p$-closed set so that V_F is $g\mu S_p$-open. Moreover, $F \subseteq V_F$. Since $V_F \subseteq g\mu S_p c_\mu(V_F) \subseteq X \backslash U_x$, $x \notin V_F$. This means that

$$x \notin \cap\{g\mu S_p c_\mu(U) : F \subseteq U \text{ and } U \text{ is } g\mu S_p\text{-open}\}.$$

Hence, $\cap\{g\mu S_p c_\mu(U) : F \subseteq U \text{ and } U \text{ is } g\mu S_p\text{-open}\} \subseteq \{x\}$. Therefore,

$$F = \cap\{g\mu S_p c_\mu(U) : F \subseteq U \text{ and } U \text{ is } g\mu S_p\text{-open}\}.$$

(v) $\Rightarrow$ (i): Let $x \notin F$ and suppose that F is a μS_p-closed set. Then $x \notin \cap\{g\mu S_p c_\mu(U) : F \subseteq U$ and U is $g\mu S_p$-open$\}$. Thus, there exists a $g\mu S_p$-open set U such that $F \subseteq U$ and $x \notin g\mu S_p c_\mu(U)$. Let $U_x = X \backslash g\mu S_p c_\mu(U)$. Then $x \in U_x$ and $U_x \cap U = \varnothing$. Therefore, X is a $g\mu S_p$-regular space. □

Theorem 4.4 *The following statements are equivalent for a GT-space X where $g\mu S_p O(X)$ possesses the property ϑ.*

(i) *X is a $g\mu S_p$-normal space.*

(ii) *For every μS_p-closed sets F_1 and F_2 such that $F_1 \cap F_2 = \varnothing$, there exist $g\mu S_p$-open sets U_1 and U_2 such that $F_1 \subseteq U_1$, $F_2 \subseteq U_2$ and $g\mu S_p c_\mu(U_1) \cap U_2 = \varnothing$.*

(iii) *For every μS_p-closed sets F_1 and F_2 such that $F_1 \cap F_2 = \varnothing$, there exists a $g\mu S_p$-open set U such that $F_1 \subseteq U$ and $g\mu S_p c_\mu(U) \cap F_2 = \varnothing$.*

(iv) *For every μS_p-closed set F and a μS_p-open set U such that $F \subseteq U$, then there exists a $g\mu S_p$-open set V such that $F \subseteq V \subseteq g\mu S_p c_\mu(V) \subseteq U$.*

Proof: (i)⇒(ii): This follows from Lemma 4.1.
(ii) ⇒ (iii): This is straightforward.
(iii)⇒ (iv): Let F be a μS_p-closed set and U be a μS_p-open set such that $F \subseteq U$. Then $F_1 = X \backslash U$ is a μS_p-closed set such that $F \cap F_1 = \varnothing$. By (iii), there exists a $g\mu S_p$-open set V such that $F \subseteq V$ and $g\mu S_p c_\mu(V) \cap F_1 = \varnothing$. Hence, $F \subseteq V \subseteq g\mu S_p c_\mu(V) \subseteq X \backslash F_1 = U$.
(iv) ⇒ (i): Suppose that F_1 and F_2 are μS_p-closed subsets of X such that $F_1 \cap F_2 = \varnothing$. Then, $U_2 = X \backslash F_2$ is a $g\mu S_p$-open set such that $F_1 \subseteq U_2$. Thus, by (iv), there exists a $g\mu S_p$-open set V such that $F_1 \subseteq V \subseteq g\mu S_p c_\mu(V) \subseteq U_2$. Hence, $F_1 \subseteq V$ and $F_2 = X \backslash U_2 \subseteq X \backslash g\mu S_p c_\mu(V)$, where $X \backslash g\mu S_p c_\mu(V)$ is a $g\mu S_p$-open set. Thus, there exist $g\mu S_p$-open sets V and $W = X \backslash g\mu S_p c_\mu(V)$ such that $F_1 \subseteq V$ and $F_2 \subseteq W$. Therefore, X is a $g\mu S_p$-normal space. □

Theorem 4.5 *The following statements are equivalent for a GT-space X where $g\mu S_p O(X)$ possesses the property ϑ.*

(i) *X is a $g\mu S_p$-T_4 space.*

(ii) *For every μS_p-closed sets F_1 and F_2 such that $F_1 \cap F_2 = \varnothing$, there exist $g\mu S_p$-open sets U_1 and U_2 such that $F_1 \subseteq U_1$, $F_2 \subseteq U_2$ and $g\mu S_p c_\mu(U_1) \cap U_2 = \varnothing$.*

(iii) *For every μS_p-closed sets F_1 and F_2 such that $F_1 \cap F_2 = \varnothing$, there exists a $g\mu S_p$-open set U such that $F_1 \subseteq U$ and $g\mu S_p c_\mu(U) \cap F_2 = \varnothing$.*

(iv) *For every μS_p-closed set F and a μS_p-open set U such that $F \subseteq U$, then there exists a $g\mu S_p$-open set V such that $F \subseteq V \subseteq g\mu S_p c_\mu(V) \subseteq U$.*

Proof: The proof is similar to Theorem 4.4. □

We will now detemine under which type of functions previously defined do some spaces are invariant.

Theorem 4.6 *The property of being a $g\mu S_p$-T_1 space and $g\mu S_p$-T_2 space are invariant under an absolute $g\mu S_p$-open bijective functions.*

Proof: Suppose that X is a $g\mu S_p$-T_1 space and Y be any space. Let $f : (X, \mu_X) \to (Y, \mu_Y)$ be an absolute $g\mu S_p$-open bijective function. Let y_1 and y_2 be any two distinct points in Y. Since f is bijective, there exist distinct points x_1 and x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is a $g\mu S_p$-T_1 space, there exist $g\mu_X S_p$-open sets U and V such that $x_1 \in U, x_2 \notin U$ and $x_2 \in V, x_1 \notin V$. Hence, $y_1 = f(x_1) \in f(U)$, $y_2 = f(x_2) \notin f(U)$ and $y_2 = f(x_2) \in f(V), y_1 = f(x_1) \notin f(V)$. Since f is an absolute $g\mu S_p$-open function, $f(U)$ and $f(V)$ are $g\mu_Y S_p$-open sets in Y

with $y_1 \in f(U), y_2 \notin f(U), y_2 \in f(V), y_1 \notin f(V)$. Therefore, Y is a $g\mu S_p$-T_1 space. The second statement is proved similarly. □

We end this section by the following theorem.

Theorem 4.7 *The property of being a $g\mu S_p$-regular space and $g\mu S_p$-normal space are invariant under absolute $g\mu S_p$-continuous and absolute $g\mu S_p$-open bijective functions.*

Proof: Let $f : (X, \mu_X) \to (Y, \mu_Y)$ be a bijective absolute $g\mu S_p$-continuous and absolute $g\mu S_p$-open function. Suppose that X is a $g\mu S_p$-regular space, $y \in Y$ and F be a $\mu_Y S_p$-closed subset of Y such that $y \notin F$. Since f is a bijective absolute $g\mu S_p$-continuous, there exists $x \in X$ such that $f(x) = y$ and $f^{-1}(F)$ is a $\mu_X S_p$-closed subset of X with $x \notin f^{-1}(F)$. Thus, there exist $g\mu_X S_p$-open sets U and V such that $x \in U$, $f^{-1}(F) \subseteq V$ and $U \cap V = \varnothing$. Hence, $y = f(x) \in f(U)$, $F \subseteq f(V)$ and $f(U) \cap f(V) = f(U \cap V) = f(\varnothing) = \varnothing$. Since f is an absolute $g\mu S_p$-open function, $f(U)$ and $f(V)$ are $g\mu_Y S_p$-open sets in Y. Therefore, Y is $g\mu S_p$-regular. The second statement is proved similarly. □

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μS_p-Sets and μS_p-Functions *

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Abstract

In this paper, the concepts of μS_p-open sets, μS_p-interior and μS_p-closure of a set in the generalized topological spaces are introduced. This also investigates related concepts such as μS_p-continuous, μS_p-open and μS_p-closed functions.

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Keywords: μ-semiopen sets, μ-preopen sets, μS_p-open sets, μS_p-functions

1 Introduction

In 1963, Levine introduced the notion of semi-open sets [3] which is one of the well-known notion of generalized open sets. Several types of generalized open sets were introduced such as preopen sets [4] that was established by Mashhour et.al in 1982. In 2007, the concept of S_p-open sets [5] in topological spaces was introduced by Shareef in his M.Sc. Thesis.

In this paper, the concepts of μS_p-open sets, μS_p-interior and μS_p-closure of a set in the generalized topological spaces are introduced and characterized. Also, the study of related functions such as μS_p-continuous, μS_p-open and μS_p-closed functions are considered.

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Throughout this paper, space (X, μ) (or simply X) always means a generalized topological space (GT-space) on which no separation axioms are assumed unless explicitly stated. For a subset A of a GT-space X, $\mu S_p c_\mu(A)$, $\mu S_p i_\mu(A)$, and $X \backslash A$ denote the μS_p-closure of A, μS_p-interior of A, and complement of A in X, respectively.

2 Preliminaries

Definition 2.1 [2] A *generalized topology* (briefly GT) on X is a subset μ of the power set $\mathscr{P}(X)$ of X such that $\varnothing \in \mu$ and every union of some elements of μ belongs to μ. We say that μ is a strong GT [1] if $X \in \mu$. The pair (X, μ) is called a generalized topological space (briefly GT-space). From now on, X will simply mean a GT-space if no confusion arises.

Definition 2.2 [1] A subset A of a GT-space X is called
(i) *μ-semiopen* if $A \subseteq c_\mu(i_\mu(A))$;
(ii) *μ-preopen* if $A \subseteq i_\mu(c_\mu(A))$;
The complement of *μ-semiopen* (respectively *μ-preopen*) set with respect to X is called a *μ-semiclosed* (respectively *μ-preclosed*) set.

Remark 2.3 A subset A of a GT-space X is
i.) *μ-semiclosed* if $i_\mu(c_\mu(A)) \subseteq A$.
ii.) *μ-preclosed* if $c_\mu(i_\mu(A)) \subseteq A$.

Definition 2.4 A subset A of a GT-space X is called *μS_p-open* if A is μ-semiopen and for every $x \in A$, there exists a μ-preclosed set F such that $x \in F \subseteq A$. The complement of a μS_p-open set is called a *μS_p-closed* set.

Remark 2.5 μ-open set and μS_p-open set are independent to each other as seen from the following example.

Example 2.6 Let $X = \{a, b, c, d\}$ and $\mu = \{\varnothing, \{a, c\}, \{d\}, \{a, c, d\}\}$. The μS_p-open sets of X are $\varnothing, X, \{b, d\}, \{a, b, c\}$, and $\{b\}$.

Remark 2.7 i.) A subset A of a GT-space X is μS_p-closed if and only if A is μ-semiclosed and for every $x \notin A$, there exists a μ-preopen set U such that $x \notin U$ and $A \subseteq U$.
ii.) The collection of all μS_p-open sets in X forms a strong GT but not always a topology on X.
iii.) The arbitrary intersection of μS_p-closed sets in X is μS_p-closed.

Definition 2.8 The union of all the μS_p-open sets of a GT-space X contained in A is called the *μS_p-interior of* A, denoted by $\mu S_p i_\mu(A)$.

Definition 2.9 The intersection of all the μS_p-closed sets of X containing A is called the μS_p-*closure* of A, denoted by $\mu S_p c_\mu(A)$.

Theorem 2.10 Let (X, μ) be a GT-space and $A \subseteq X$. Then the following hold:
i.) $\mu S_p i_\mu(A) = X \backslash (\mu S_p c_\mu(X \backslash A))$.
ii.) $\mu S_p c_\mu(A) = X \backslash \mu S_p i_\mu(X \backslash A)$.

Definition 2.11 Let (X, μ_X) and (Y, μ_Y) be GT-spaces. A function $f : (X, \mu_X) \to (Y, \mu_Y)$ is called
(i) μ-*continuous* if for every μ_Y-open subset U of Y, $f^{-1}(U)$ is μ_X-open in X.
(ii) μS_p-*continuous* if for every μ_Y-open subset U of Y, $f^{-1}(U)$ is $\mu_X S_p$-open in X.
(iii) μS_p-*regular strongly continuous* (briefly $\mu S_p rs$-continuous) if the inverse image of every $\mu_Y S_p$-open set in Y is μ_X-open in X.
(iv) μS_p-*open* if the image $f(A)$ is $\mu_Y S_p$-open in Y for each μ_X-open set A in X.
(v) μS_p-*closed* if the image $f(A)$ is $\mu_Y S_p$-closed in Y for each μ_X-closed set A in X.
(vi) μS_p-irresolute if for every $\mu_Y S_p$-closed subset F of Y, $f^{-1}(F)$ is a $\mu_X S_p$-closed set in X.

3 μS_p-interior and μS_p-closure of a Set

Theorem 3.1 *Let (X, μ) be a GT-space and $A, B \subseteq X$. Then*

(a) *A is μS_p-open if and only if $A = \mu S_p i_\mu(A)$.*

(b) *If $A \subseteq B$, then $\mu S_p i_\mu(A) \subseteq \mu S_p i_\mu(B)$.*

(c) *If A and B are both μS_p-open, then $A \cap B$ is not necessarily a μS_p-open set.*

Remark 3.2 The collection of all μS_p-open sets in X does not necessarily form a topological space.

To see this, consider $X = \{a, b, c, d\}$ with $\mu = \mathscr{P}(X)$. Then the μS_p-open sets in X are $\varnothing, X, \{a, c, d\}$ and $\{b, c, d\}$ but $\{a, c, d\} \cap \{b, c, d\} = \{c, d\}$ is not μS_p-open.

Remark 3.3 Let (X, μ) be a GT-space and $A \subseteq X$.
i.) If A is μ-open, then A is both μ-semiopen and μ-preopen.
ii.) If A is μ-closed, then A is both μ-semiclosed and μ-preclosed.

Theorem 3.4 *Let (X, μ) be a GT-space and $A \subseteq X$. If A is both μ-semiopen and μ-preclosed, then A is μS_p-open.*

The converse of Theorem 3.4 is not true since in Example 2.6, $\varnothing$ is μS_p-open but it is not μ-preclosed.

Theorem 3.5 *Let (X, μ) be a GT-space and $A, B \subseteq X$. Then*

(a) *$x \in \mu S_p c_\mu(A)$ if and only if for every μS_p-open set O with $x \in O$, $O \cap A \neq \varnothing$.*

(b) *If $A \subseteq B$, then $\mu S_p c_\mu(A) \subseteq \mu S_p c_\mu(B)$.*

(c) *$\mu S_p c_\mu(A) \subseteq \mu S_p c_\mu(\mu S_p c_\mu(A))$.*

(d) *A is μS_p-closed if and only if $A = \mu S_p c_\mu(A) = \mu S_p c_\mu(\mu S_p c_\mu(A))$.*

(e) *$\mu S_p c_\mu(A) \cup \mu S_p c_\mu(B) \subseteq \mu S_p c_\mu(A \cup B)$.*

Theorem 3.6 *Let (X, μ) be a GT-space. The intersection of all the μ-closed subsets A_i of X is a μS_p-open set.*

Proof: Let $\{A_i : i \in I\}$ be a collection of all μ-closed subsets of X. If $\cap_i A_i = \varnothing$, then we are done. Assume that $\cap_i A_i \neq \varnothing$. Since $A_i's$ are μ-closed sets, $\cap_i A_i$ is μ-closed. We claim that $i_\mu(\cap_i A_i) = \varnothing$. Indeed, if in contrary, $i_\mu(\cap_i A_i) \neq \varnothing$, then there exists a μ-open set $B \neq \varnothing$ such that $B \subseteq \cap_i A_i$. Thus, $X \backslash B$ is μ-closed and since $\cap_i A_i$ is the smallest μ-closed subset of X, $\cap_i A_i \subseteq X \backslash B$. This implies that $B \subseteq X \backslash B$, a contradiction. This shows that $i_\mu(\cap_i A_i) = \varnothing$. Hence, $c_\mu(i_\mu(\cap_i A_i)) = c_\mu(\varnothing) = \cap_i A_i$ so $\cap_i A_i$ is μ-semiopen and μ-preclosed. Therefore, $\cap_i A_i$ is μS_p-open. □

Theorem 3.7 *Let X be a finite nonempty set and $\mu_x = \mathscr{P}(X \backslash \{x\})$ where $x \in X$. Then μ_x is a GT on X and every μ_x-closed set is a $\mu_x S_p$-open set.*

Proof: Let $x \in X$ and A be a μ_x-closed set. Then $A \subseteq X$ with $x \in A$. By Remark 3.3, A is a μ_x-preclosed set. We claim that A is μ_x-semiopen. If $A = \{x\}$, then $i_{\mu_x}(A) = \varnothing$ and $c_{\mu_x}(i_{\mu_x}(A)) = c_{\mu_x}(\varnothing) = \{x\} = A$. Hence, A is μ_x-semiopen. Suppose that $|A| > 1$. Then there exists $y \in A$ such that $x \neq y$. Then $y \in i_{\mu_x}(A)$. Thus, every μ_x-closed set $F \supseteq i_{\mu_x}(A)$ contains y. Hence, $y \in c_{\mu_x}(i_{\mu_x}(A))$. Therefore, $A \subseteq c_{\mu_x}(i_{\mu_x}(A))$. It follows that A is μ_x-semiopen. By Theorem 3.4, A is a $\mu_x S_p$-open set. □

4 μS_p-continuous Functions

In this section, some properties of μS_p-continuous functions are obtained.

Theorem 4.1 *If $f : (X, \mu_X) \to (Y, \mu_Y)$ is μS_p-continuous and $g : (Y, \mu_Y) \to (Z, \mu_Z)$ is μ-continuous, then $g \circ f : (X, \mu_X) \to (Z, \mu_Z)$ is μS_p-continuous.*

Proof: Let U be μ_Z-open in Z. Then $g^{-1}(U)$ is μ_Y-open since g is μ-continuous. Thus, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is $\mu_X S_p$-open since f is μS_p-continuous. Therefore, $g \circ f$ is μS_p-continuous. □

Remark 4.2 *The composition of two μS_p-continuous functions need not be μS_p-continuous.*

To see this, let $X = \{a, b, c, d\}$ with $\mu_X = \{\varnothing, \{a, c\}, \{d\}, \{a, c, d\}\}$, $Y = \{a, b\}$ with $\mu_Y = \{\varnothing, \{a\}\}$, and $Z = \{u, v, w\}$ with $\mu_Z = \{\varnothing, \{u\}\}$. Then $\mu_Y S_p$-open sets in Y are $\varnothing, \{a\}, \{b\}, \{a, b\}$; $\mu_X S_p$-open sets in X are $\varnothing, X, \{b, d\}, \{a, b, c\}, \{b\}$; $\mu_Z S_p$-open sets in Z are $\varnothing, Z, \{v, w\}$. Define $f : X \to Y$ by $f(b) = a$, $f(a) = f(c) = f(d) = b$. Then f is μS_p-continuous. Define $g : Y \to Z$ by $g(b) = u$, $g(a) = v$. Then g is also μS_p-continuous. But $g \circ f : X \to Z$ and $(g \circ f)^{-1}(\{u\}) = f^{-1}(g^{-1}(\{u\})) = f^{-1}(\{b\}) = \{a, c, d\}$ is not a $\mu_X S_p$-open set in X.

Theorem 4.3 *Let $f : (X, \mu_X) \to (Y, \mu_Y)$ be a bijective function. The following statements are equivalent:*
1. f is μS_p-continuous.
2. For each $x \in X$, and each μ_Y-open set V containing $f(x)$, there exists a $\mu_X S_p$-open set U containing x such that $f(U) \subseteq V$.
3. $f^{-1}(F)$ is $\mu_X S_p$-closed in X for every μ_Y-closed set F in Y.
4. $f(\mu_X S_p c_{\mu_X}(A)) \subseteq c_{\mu_Y}(f(A))$ for every $A \subseteq X$.
5. $\mu_X S_p c_{\mu_X}(f^{-1}(B)) \subseteq f^{-1}(c_{\mu_Y}(B))$ for every $B \subseteq Y$.
6. $f^{-1}(i_{\mu_Y}(B)) \subseteq \mu_X S_p i_{\mu_X}(f^{-1}(B))$ for every $B \subseteq Y$.
7. $i_{\mu_Y}(f(A)) \subseteq f(\mu_X S_p i_{\mu_X}(A))$ for every subset A of X.

Proof: (1) ⇒ (2): Let $x \in X$ and let V be a μ_Y-open set with $f(x) \in V$. Since f is μS_p-continuous, $f^{-1}(V)$ is $\mu_X S_p$-open in X and $x \in f^{-1}(V)$. Take $U = f^{-1}(V)$ so that $f(U) = V$ with $x \in U$.
(2) ⇒ (1): Let V be any μ_Y-open set in Y and let $x \in f^{-1}(V)$. Then $f(x) \in V$. By (2), there exists a $\mu_X S_p$-open set U_x such that $x \in U_x$ and $f(U_x) \subseteq V$. By Remark 2.7 (ii), $\bigcup_{x \in f^{-1}(V)} U_x$ is a $\mu_X S_p$-open set in X. Hence, $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ is a $\mu_X S_p$-open set. Therefore, f is μS_p-continuous.

(1) $\Leftrightarrow$ (3): Let f be μS_p-continuous and let F be any μ_Y-closed set in Y. Then $Y\backslash F$ is μ_Y-open. Since f is μS_p-continuous, $f^{-1}(Y\backslash F)$ is $\mu_X S_p$-open. Now, $f^{-1}(Y\backslash F) = f^{-1}(Y)\backslash f^{-1}(F) = X\backslash f^{-1}(F)$. Hence, $f^{-1}(F)$ is $\mu_X S_p$-closed in X. Conversely, let F be a μ_Y-open set in Y. Then $Y\backslash F$ is μ_Y-closed. By assumption, $f^{-1}(Y\backslash F)$ is $\mu_X S_p$-closed in X. Since $f^{-1}(Y\backslash F) = X\backslash f^{-1}(F)$, $f^{-1}(F)$ is $\mu_X S_p$-open. Therefore, f is μS_p-continuous.
(3) $\Rightarrow$ (4): Let A be any subset of X. Then $f(A) \subseteq c_{\mu_Y}(f(A))$ and $c_{\mu_Y}(f(A))$ is a μ_Y-closed set in Y. By assumption, $f^{-1}(c_{\mu_Y}(f(A)))$ is a $\mu_X S_p$-closed set in X. Hence, $\mu S_p c_{\mu_X}(A) \subseteq f^{-1}(c_{\mu_Y}(f(A))$. Therefore, $f(\mu_X S_p c_{\mu_X}(A)) \subseteq c_{\mu_Y}(f(A))$.
(4) $\Rightarrow$ (5): Let $B \subseteq Y$. Then $f^{-1}(B)$ is a subset of X. By (4), $f(\mu_X S_p c_{\mu_X}(f^{-1}(B)) \subseteq c_{\mu_Y} f(f^{-1}(B)) \subseteq c_{\mu_Y}(B)$. Thus, $\mu_X S_p c_{\mu_X}(f^{-1}(B)) \subseteq f^{-1}(c_{\mu_Y}(B))$.
(5) $\Rightarrow$ (6): Let $B \subseteq Y$. Applying (5) to $Y\backslash B$, we have $\mu_X S_p c_{\mu_X}(f^{-1}(Y\backslash B)) \subseteq f^{-1}(c_{\mu_Y}(Y\backslash B))$. It follows that $f^{-1}(i_{\mu_Y}(B)) \subseteq \mu_X S_p i_{\mu_X}(f^{-1}(B))$.
(6) $\Rightarrow$ (7): Let A be any subset of X. Then $f(A)$ is in Y. By (6), $f^{-1}(i_{\mu_Y}(f(A))) \subseteq \mu_X S_p i_{\mu_X}(A)$. Therefore, $i_{\mu_Y}(f(A)) \subseteq f(\mu_X S_p i_{\mu_X}(A))$.
(7) $\Rightarrow$ (1): Let V be a μ_Y-open subset of Y. Then $f^{-1}(V) \subseteq X$. By (7), $i_{\mu_Y}(f(f^{-1}(V))) \subseteq f(\mu_X S_p i_{\mu_X}(f^{-1}(V)))$. Thus, $i_{\mu_Y}(V) \subseteq f(\mu_X S_p i_{\mu_X}(f^{-1}(V)))$. Since V is μ_Y-open, $V \subseteq f(\mu_X S_p i_{\mu_X}(f^{-1}(V)))$ so that $f^{-1}(V) \subseteq \mu_X S_p i_{\mu_X}(f^{-1}(V))$. Hence, $\mu_X S_p i_{\mu_X}(f^{-1}(V)) = f^{-1}(V)$ which is $\mu_X S_p$-open. Therefore, f is μS_p-continuous. The proof is complete. □

Theorem 4.4 *$f : X \to Y$ is $\mu S_p rs$-continuous if and only if $f^{-1}(A)$ is μ-closed for every μS_p-closed set A in Y.*

Proof: Let f be a $\mu S_p rs$-continuous function and A be a $\mu_Y S_p$-closed set in Y. Then $Y\backslash A$ is $\mu_Y S_p$-open in Y. Thus, $f^{-1}(Y\backslash A)$ is μ_X-open since f is $\mu S_p rs$-continuous. But $f^{-1}(Y\backslash A) = X\backslash f^{-1}(A)$. Hence, $f^{-1}(A)$ is μ-closed.

Conversely, let O be a $\mu_Y S_p$-open set in Y. Then $Y\backslash O$ is $\mu_Y S_p$-closed. By assumption, $f^{-1}(Y\backslash O)$ is μ_X-closed. Thus, $f^{-1}(Y\backslash O) = X\backslash f^{-1}(O)$ is μ_X-closed. Therefore, $f^{-1}(O)$ is μ_X-open implying that f is $\mu S_p rs$-continuous. This proves the theorem. □

Theorem 4.5 *Let $f : (X, \mu_X) \to (Y, \mu_Y)$ be a bijective function. The following statements are equivalent:*
1. f is $\mu S_p rs$-continuous.
2. For each $x \in X$, and each $\mu_Y S_p$-open set V containing $f(x)$, there exists a μ_X-open set U containing x such that $f(U) \subseteq V$.
3. $f^{-1}(F)$ is μ_X-closed in X for every $\mu_Y S_p$-closed set F in Y.
4. $f(c_{\mu_X}(A)) \subseteq \mu_Y S_p c_{\mu_Y}(f(A))$ for every $A \subseteq X$.
5. $c_{\mu_X}(f^{-1}(B)) \subseteq f^{-1}(\mu_X S_p c_{\mu_X}(B))$ for every $B \subseteq Y$.
6. $f^{-1}(\mu_Y S_p i_{\mu_Y}(B)) \subseteq i_{\mu_X}(f^{-1}(B))$ for every $B \subseteq Y$.
7. $\mu_Y S_p i_{\mu_Y}(f(A)) \subseteq f(i_{\mu_X}(A))$ for every subset A of X.

Proof: The proof is analogous to Theorem 4.3. □

5 μS_p-open and μS_p-closed Functions

This section includes some properties of μS_p-open and μS_p-closed functions.

Theorem 5.1 *Let $f : (X, \mu_X) \to (Y, \mu_Y)$ be a bijective function. Then the following statements are equivalent:*

1. *f is μS_p-open.*
2. *f is μS_p-closed.*
3. *$f(i_{\mu_X}(A)) \subseteq \mu_Y S_p i_{\mu_Y}(f(A))$ for every $A \subseteq X$.*
4. *For each subset W of Y and each μ_X-open set U containing $f^{-1}(W)$, there exists a $\mu_Y S_p$-open set V of Y such that $W \subseteq V$ and $f^{-1}(V) \subseteq U$.*
5. *For every subset S of Y and for every μ_X-closed set F of X containing $f^{-1}(S)$, there exists a $\mu_Y S_p$-closed set K of Y containing S such that $f^{-1}(K) \subseteq F$.*
6. *$f^{-1}(\mu_Y S_p c_{\mu_Y}(B)) \subseteq c_{\mu_X}(f^{-1}(B))$ for every subset B of Y.*
7. *$\mu_Y S_p c_{\mu_Y}(f(A)) \subseteq f(c_{\mu_X}(A))$ for every subset A of X.*

Proof:
(1) ⇔ (2): Let f be μS_p-open and D be μ_X-closed in X. Then $X \backslash D$ is μ_X-open and $f(X \backslash D)$ is $\mu_Y S_p$-open. Since f is bijective, $Y \backslash f(D) = f(X \backslash D)$ is $\mu_Y S_p$-open. Thus, $f(D)$ is $\mu_Y S_p$-closed.

Conversely, let f be μS_p-closed and suppose that O is a μ_X-open set in X. Then $X \backslash O$ is μ_X-closed and $f(X \backslash O) = Y \backslash f(O)$ is $\mu_Y S_p$-closed. Therefore, $f(O)$ is $\mu_Y S_p$-open.

(1) ⇔ (3): Let $A \subseteq X$ and suppose that f is μS_p-open. Since $i_{\mu_X}(A)$ is μ_X-open and f is μS_p-open, $f(i_{\mu_X}(A))$ is $\mu_Y S_p$-open. Also, $i_{\mu_X}(A) \subseteq A$ implies that $f(i_{\mu_X}(A)) \subseteq f(A)$. Thus, $f(i_{\mu_X}(A)) \subseteq \mu_Y S_p i_{\mu_Y}(f(A))$ by definition of $\mu_Y S_p i_{\mu_Y}(f(A))$.

Conversely, let O be a μ_X-open set in X. Then $i_{\mu_X}(O) = O$ and $f(i_{\mu_X}(O)) = f(O) \subseteq \mu_Y S_p i_{\mu_Y}(f(O)) \subseteq f(O)$. Hence, $\mu_Y S_p i_{\mu_Y}(f(O)) = f(O)$. Since $\mu_Y S_p i_{\mu_Y}(f(O))$ is $\mu_Y S_p$-open, $f(O)$ is $\mu_Y S_p$-open. Therefore, f is a μS_p-open function.

(2) ⇔ (7): Let $A \subseteq X$ and suppose that f is μS_p-closed. Since $A \subseteq c_{\mu_X}(A)$, $f(A) \subseteq f(c_{\mu_X}(A))$. Moreover, since $c_{\mu_X}(A)$ is μ_X-closed in X, $f(c_{\mu_X}(A))$ is $\mu_Y S_p$-closed. Therefore, $\mu_Y S_p c_{\mu_Y}(f(A)) \subseteq f(c_{\mu_X}(A))$.

Conversely, let O be μ_X-closed. Then $c_{\mu_X}(O) = O$ and $f(c_{\mu_X}(O)) = f(O)$. Since $f(O) \subseteq \mu_Y S_p c_{\mu_Y}(f(O)) \subseteq f(c_{\mu_X}(O)) = f(O)$, $\mu_Y S_p c_{\mu_Y}(f(O)) = f(O)$. Since $\mu_Y S_p c_{\mu_Y}(f(O))$ is $\mu_Y S_p$-closed, $f(O)$ is $\mu_Y S_p$-closed. Therefore, f is a

μS_p-closed function.
(1) $\Leftrightarrow$ (5): Suppose that f is μS_p-open. Let $S \subseteq Y$ and F be a μ_X-closed subset of X such that $f^{-1}(S) \subseteq F$. Now, $X\backslash F$ is a μ_X-open set in X. Since f is μS_p-open, $f(X\backslash F)$ is $\mu_Y S_p$-open in Y. Then $K = Y\backslash f(X\backslash F)$ is a $\mu_Y S_p$-closed set in Y. Since $f^{-1}(S) \subseteq F$, $X\backslash F \subseteq X\backslash f^{-1}(S) = f^{-1}(Y\backslash S)$. Thus, $f(X\backslash F) \subseteq f(f^{-1}(Y\backslash S)) \subseteq Y\backslash S$. Hence $Y\backslash(Y\backslash S) \subseteq Y\backslash f(X\backslash F)$ implying that $S \subseteq K$ and $f^{-1}(K) = X\backslash f^{-1}(f(X\backslash F)) \subseteq X\backslash(X\backslash F) = F$.

For the converse, let U be a μ_X-open set in X. Since $X\backslash U$ is μ_X-closed and $f^{-1}(Y\backslash f(U)) = X\backslash(f^{-1}(f(U))) \subseteq X\backslash U$, by assumption, there exists a $\mu_Y S_p$-closed subset K of Y such that $Y\backslash f(U) \subseteq K$ and $f^{-1}(K) \subseteq X\backslash U$ so that $U \subseteq X\backslash f^{-1}(K)$. Hence, $Y\backslash K \subseteq f(U) \subseteq f(X\backslash f^{-1}(K)) \subseteq Y\backslash K$. This implies that $f(U) = Y\backslash K$. Since $Y\backslash K$ is $\mu_Y S_p$-open, $f(U)$ is $\mu_Y S_p$-open in Y. Therefore, f is μS_p-open.
(2) $\Leftrightarrow$ (4): Similar to (1) $\Leftrightarrow$ (5).
(1) $\Leftrightarrow$ (6): Suppose that $f : X \to Y$ is a μS_p-open function and let B be any subset of Y. Since $f^{-1}(B) \subseteq c_{\mu_X}(f^{-1}(B))$ and $c_{\mu_X}(f^{-1}(B))$ is μ_X-closed in X, by (1) $\Leftrightarrow$ (5), there exists a $\mu_Y S_p$-closed set K of Y such that $B \subseteq K$ and $f^{-1}(K) \subseteq c_{\mu_X}(f^{-1}(B))$. Hence, $\mu_Y S_p c_{\mu_Y}(B) \subseteq K$. Therefore, $f^{-1}(\mu_Y S_p c_{\mu_Y}(B)) \subseteq f^{-1}(K) \subseteq c_{\mu_X}(f^{-1}(B))$.

Conversely, let O be a μ_X-open set in X. Then $X\backslash O$ is μ_X-closed and $f^{-1}(\mu_Y S_p c_{\mu_Y}(f(X\backslash O))) \subseteq X\backslash O$. Also, $X\backslash O \subseteq f^{-1}(\mu_Y S_p c_{\mu_Y}(f(X\backslash O)))$ and $\mu_Y S_p c_{\mu_Y}(f(X\backslash O)) = Y\backslash f(O)$. Since $\mu_Y S_p c_{\mu_Y}(f(X\backslash O))$ is $\mu_Y S_p$-closed, $f(O)$ is $\mu_Y S_p$-open. Therefore, f is a μS_p-open function. □

Theorem 5.2 *Let $f : (X, \mu_X) \to (Y, \mu_Y)$ and $g : (Y, \mu_Y) \to (Z, \mu_Z)$ be mappings such that the composition $g \circ f : (X, \mu_X) \to (Z, \mu_Z)$ is μS_p-closed. Then the following hold:*

(a) *If f is μ-continuous and surjective, then g is μS_p-closed.*

(b) *If g is μS_p-irresolute and injective, then f is μS_p-closed.*

(c) *If g is $\mu S_p rs$-continuous and injective, then f is μ-closed.*

Proof: (a) Let f be μ-continuous and surjective and let A be a μ_Y-closed subset of Y. Since f is μ-continuous, $f^{-1}(A)$ is μ_X-closed in X. Since $g \circ f$ is μS_p-closed, $(g \circ f)(f^{-1}(A))$ is $\mu_Z S_p$-closed in Z. Since f is surjective, $(g \circ f)(f^{-1}(A)) = g(f(f^{-1}(A))) = g(A)$ is also $\mu_Z S_p$-closed. Therefore, $g(A)$ is a $\mu_Z S_p$-closed set in Z and g is a μS_p-closed function.
(b) Let $A \subseteq X$ be a μ_X-closed set. Since $g \circ f$ is μS_p-closed, $(g \circ f)(A)$ is $\mu_Z S_p$-closed in Z. Because g is μS_p-irresolute and injective, $f(A) = g^{-1}(g(f(A))) = g^{-1}((g \circ f)(A))$ is $\mu_Y S_p$-closed in Y. Therefore, f is μS_p-closed.

(c) Let D be a μ_X-closed set of X. Since $g \circ f$ is μS_p-closed, $(g \circ f)(D)$ is $\mu_Z S_p$-closed in Z. Since g is $\mu S_p rs$-continuous and injective, $f(D) = g^{-1}((g \circ f)(D))$ is μ_Y-closed in Y. That is, $f(D)$ is μ_Y-closed in Y. Therefore, f is μ-closed. This completes the proof. □

Theorem 5.3 *Let $f : (X, \mu_X) \to (Y, \mu_Y)$ be a μ-closed map and $g : (Y, \mu_Y) \to (Z, \mu_Z)$ a μS_p-closed map, then the composition $g \circ f : (X, \mu_X) \to (Z, \mu_Z)$ is μS_p-closed.*

Proof: Let F be any μ_X-closed set in X. Since f is μ-closed, $f(F)$ is μ_Y-closed in Y. Because g is μS_p-closed, $g(f(F))$ is $\mu_Z S_p$-closed in Z. Thus, $(g \circ f)(F) = g(f(F))$ is $\mu_Z S_p$-closed and hence $g \circ f$ is μS_p-closed. □

Remark 5.4 *Let $f : (X, \mu_X) \to (Y, \mu_Y)$ be a μS_p-closed function and $g : (Y, \mu_Y) \to (Z, \mu_Z)$ a μ-closed function. Then the composition $g \circ f : (X, \mu_X) \to (Z, \mu_Z)$ need not be μS_p-closed.*

Theorem 5.5 *For a bijection map $f : (X, \mu_X) \to (Y, \mu_Y)$, the following are equivalent:*

(a) *$f^{-1} : Y \to X$ is μS_p-continuous.*

(b) *f is μS_p-open.*

(c) *f is μS_p-closed.*

Proof: (a)⇒(b): Let U be a μ_X-open set of X. By hypothesis, $(f^{-1})^{-1}(U) = f(U)$ is $\mu_Y S_p$-open in Y so that f is μS_p-open.
(b)⇒(c): Let F be a μ_X-closed set of X. Then $X \backslash F$ is μ_X-open in X. By assumption, $f(X \backslash F)$ is $\mu_Y S_p$-open in Y. Since f is bijective, $Y \backslash f(F) = f(X \backslash F)$ is $\mu_Y S_p$-open in Y. Hence, $f(F)$ is $\mu_Y S_p$-closed in Y. Therefore, f is μS_p-closed.
(c)⇒(a): Let F be a μ_X-closed set of X. By (c), $f(F)$ is $\mu_Y S_p$-closed in Y. But $f(F) = (f^{-1})^{-1}(F)$. Thus, f^{-1} is μS_p-continuous. □

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Some Properties of rw-Sets and rw-Continuous Functions[1]

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Abstract

In this paper, the concept of regular w-closed (rw-closed) sets in topological spaces introduced in [1] is further studied. It also investigates related concepts such as rw-interior and rw-closure of a set, and rw-continuous.

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Keywords: regular open sets, rw-sets, rw-functions

1 Introduction

In 1937, Stone [6] introduced and investigated the regular open sets. These sets are contained in the family of open sets since a set is regular open if it is equal to the interior of its closure. In 1978, Cameron [2] also introduced and investigated the concept of a regular semiopen set. A set A is regular semiopen if there is a regular open set U such that $U \subseteq A \subseteq \overline{U}$. In 2007, a new class of sets called regular w-closed sets (rw-closed sets) was introduced by Benchalli and Wali [1]. A set B is rw-closed if $\overline{B} \subseteq U$ whenever $B \subseteq U$ for any regular semiopen set U. They proved that this new class of sets is properly placed

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in between the class of w-closed sets [5] and the class of regular generalized closed sets [4].

In this paper, the concepts of rw-closed and rw-open sets (complement of rw-closed set) are further investigated. Also, the study of related functions involving rw-closed and rw-open sets are characterized.

Throughout this paper, space (X, T) (or simply X) always means a topological space on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X, $\overline{A}$, $int(A)$, and $\mathrm{C}(A)$ denote the closure of A, interior of A, and complement of A in X, respectively.

2 Preliminaries

Definition 2.1 [1] A subset A of a space X is called
(i) *regular open* if $int(\overline{A}) = A$ and it is *regular closed* if $\overline{int(A)} = A$.
(ii) *regular semiopen* if there exists a regular open set U such that $U \subseteq A \subseteq \overline{U}$.
(iii) *regular w-closed set* (briefly, rw-closed) if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and U is regular semiopen in X. The complement of any rw-closed set is called *rw-open set.*

Definition 2.2 [3] The intersection of all the rw-closed sets of X containing A is called the *rw-closure* of A, denoted by rw-$(\overline{A})$.

Definition 2.3 [3] The union of all the rw-open sets of a space X contained in A is called the *rw-interior of* A, denoted by rw-$int(A)$.

Definition 2.4 [1] A function $f : X \to Y$ is called
(i) *rw-open* if the image $f(A)$ is rw-open in Y for each open set A in X.
(ii) *rw-closed* if the image $f(A)$ is rw-closed for each closed set A in X.
(iii) *rw-continuous* if for every open subset U of Y, $f^{-1}(U)$ is rw-open in X.

Theorem 2.5 [1] *Every closed set is rw-closed.*

3 rw-interior and rw-closure of a Set

Theorem 3.1 *Let* (X, T) *be a topological space and* $A, B \subseteq X$. *Then*

(a) *If A is open, then A is rw-open.*

(b) *If A is rw-open, then $A = rw$-$int(A)$.*

(c) *$int(A) \subseteq rw\text{-}int(A)$.*

(d) *If $A \subseteq B$, then $rw\text{-}int(A) \subseteq rw\text{-}int(B)$.*

(e) *If A and B are both rw-open, then $A \cap B$ is rw-open.*

Remark 3.2 *The converses of Theorem* 3.1 (a) *and* (b) *are not true.*

Remark 3.3 *Let (X, T) be a topological space and $A, B \subseteq X$. If A and B are both rw-open, then $A \cup B$ need not be rw-open. Thus, the family of all the rw-open subsets of X is not a topology in X.*

Theorem 3.4 *A is rw-open in X if and only if for every regular semiopen set U in X with $A \cup U = X$, $int(A) \cup U = X$.*

Proof: ($\Rightarrow$) Let A be an rw-open set in X and let U be a regular semiopen with $A \cup U = X$. Then $\mathrm{C}(A) \cap \mathrm{C}(U) = \varnothing$ implying that $\mathrm{C}(A) \subseteq U$. Since $\mathrm{C}(A)$ is rw-closed, $\overline{\mathrm{C}(A)} \subseteq U$. Hence $\mathrm{C}(U) \subseteq \mathrm{C}(\overline{\mathrm{C}(A)})$. But $\mathrm{C}(\overline{\mathrm{C}(A)}) = int(A)$. Thus $\mathrm{C}(U) \subseteq int(A)$. Therefore, $int(A) \cup U = X$.
($\Leftarrow$) Let U be a regular semiopen set such that $\mathrm{C}(A) \subseteq U$. Then $\mathrm{C}(A) \cap \mathrm{C}(U) = \varnothing$ implying that $A \cup U = X$. By hypothesis, $int(A) \cup U = X$ implies that $\mathrm{C}(U) \subseteq int(A) = \mathrm{C}(\overline{\mathrm{C}(A)})$ so that $\overline{\mathrm{C}(A)} \subseteq U$. Thus $\mathrm{C}(A)$ is rw-closed. Consequently, A is rw-open. □

Theorem 3.5 *Let (X, T) be a topological space and $A, B \subseteq X$. Then*

(a) *$x \in rw\text{-}(\overline{A})$ if and only if for every rw-open set O with $x \in O$, $O \cap A \neq \varnothing$.*

(b) *For any set A, $rw\text{-}(\overline{A}) \subseteq rw\text{-}(\overline{rw\text{-}(\overline{A})})$.*

(c) *If A is rw-closed, then $A = rw\text{-}(\overline{A}) = rw\text{-}(\overline{rw\text{-}(\overline{A})})$.*

(d) *$rw\text{-}(\overline{A \cup B}) = rw\text{-}(\overline{A}) \cup rw\text{-}(\overline{B})$.*

(e) *$rw\text{-}(\overline{A}) \subseteq \overline{A}$.*

(f) *If A and B are subsets of X with $A \subseteq B$, then $rw\text{-}(\overline{A}) \subseteq rw\text{-}(\overline{B})$.*

4 rw-continuous Functions

Theorem 4.1 *Every continuous function is rw-continuous.*

Proof: Let X and Y be topological spaces and let $f : X \to Y$ be a function. Suppose that A is any open set in Y. Since f is continuous, $f^{-1}(A)$ is open in X. By Theorem 3.1(a), $f^{-1}(A)$ is rw-open. Thus, f is rw-continuous. □

Theorem 4.2 *If $f : X \to Y$ is rw-continuous and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is rw-continuous.*

Proof: Let U be open in Z. Then $g^{-1}(U)$ is open since g is continuous. Thus, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is rw-open since f is rw-continuous. Therefore, $g \circ f$ is rw-continous. □

Remark 4.3 *The composition of two rw-continuous functions need not be rw-continuous.*

Theorem 4.4 *Let X and Y be topological spaces and $f : X \to Y$. Then f is rw-continuous if and only if the inverse image of each closed set in Y is rw-closed in X.*

Proof: Let f be rw-continuous and let U be any closed set in Y. Then $Y \backslash U$ is open. Since f is rw-continuous, $f^{-1}(Y \backslash U)$ is rw-open. Now,

$$f^{-1}(Y \backslash U) = f^{-1}(Y) \backslash f^{-1}(U) = X \backslash f^{-1}(U).$$

Hence, $f^{-1}(U)$ is rw-closed in X.

Conversely, let U be open in Y. Then $Y \backslash U$ is closed. By assumption, $f^{-1}(Y \backslash U)$ is rw-closed in X. Now,

$$f^{-1}(Y \backslash U) = f^{-1}(Y) \backslash f^{-1}(U) = X \backslash f^{-1}(U).$$

Hence, $f^{-1}(U)$ is rw-open. Therefore, f is rw-continuous. □

Theorem 4.5 *If $f : X \to Y$ is rw-continuous, then $f(rw\text{-}(\overline{A})) \subseteq \overline{f(A)}$ for every $A \subseteq X$.*

Proof: Let $A \subseteq X$ and let $x \in rw\text{-}(\overline{A})$. Suppose further that U is an open set in Y with $f(x) \in U$. Since f is rw-continuous, $f^{-1}(U)$ is rw-open in X with $x \in f^{-1}(U)$. Hence, by Theorem 3.5(a), $f^{-1}(U) \cap A \neq \varnothing$. It follows that

$$\varnothing \neq f(f^{-1}(U) \cap A) \subseteq f(f^{-1}(U)) \cap f(A) \subseteq U \cap f(A).$$

Thus, $U \cap f(A) \neq \varnothing$. Hence, $f(x) \in \overline{f(A)}$. □

Theorem 4.6 *If $f : X \to Y$ is rw-continuous, then $rw\text{-}(\overline{f^{-1}(B)}) \subseteq f^{-1}(\overline{B})$ for every $B \subseteq Y$.*

Proof: Let $f : X \to Y$ be rw-continuous. Suppose that $B \subseteq Y$ and $A = f^{-1}(B)$. Then by Theorem 4.5, $f(rw\text{-}(\overline{f^{-1}(B)})) \subseteq \overline{f(f^{-1}(B))} \subseteq \overline{B}$. Thus, $rw\text{-}(\overline{f^{-1}(B)}) \subseteq f^{-1}(\overline{B})$. □

Definition 4.7 A function $f : X \to Y$ is called *regular strongly continuous* (briefly rs-continuous) if the inverse image of every rw-open set in Y is open in X, that is, $f^{-1}(A)$ is open in X for all rw-open sets A in Y.

Remark 4.8 *Every rs-continuous function is rw-continuous.*

Theorem 4.9 *$f : X \to Y$ is rs-continuous if and only if $f^{-1}(A)$ is closed for every rw-closed set A in X.*

Proof: ($\Rightarrow$) Let f be rs-continuous and let A be rw-closed in Y. Then $\mathrm{C}(A)$ is rw-open in Y. Thus, $f^{-1}(\mathrm{C}(A))$ is open since f is rs-continuous. But $f^{-1}(\mathrm{C}(A)) = \mathrm{C}(f^{-1}(A))$. Hence, $f^{-1}(A)$ is closed.
($\Leftarrow$) Let O be rw-open in Y. Then $\mathrm{C}(O)$ is rw-closed. By assumption, $f^{-1}(\mathrm{C}(O))$ is closed. Thus, $f^{-1}(\mathrm{C}(O)) = \mathrm{C}(f^{-1}(O))$ is closed. Therefore, $f^{-1}(O)$ is open implying that f is rs-continuous. □

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Vertex-Weighted (k_1, k_2) E-Torsion Graph of Quasi Self-Dual Codes

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Abstract. In this paper, we have introduced a graph G_{EC} generated by type-(k_1, k_2) E-codes which is (k_1, k_2) E-torsion graph. The binary codewords of the torsion code of C are the set of vertices, and the edges are defined using the construction of E-codes. Moreover, we characterized the graph obtained when $k_1 = 0$ and $k_2 = 0$ and calculated the degrees of every vertex and the number of edges of G_{EC}. Moreover, we presented necessary and sufficient conditions for a vertex to be in the center of a graph given the property of the codeword corresponding to the vertex. Finally, we represent every quasi self-dual codes of short length by defining the vertex-weighted (k_1, k_2) E-torsion graph, where the weight of every vertex is the weight of the codeword corresponding to the vertex.

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Key Words and Phrases: quasi-self dual codes, rings, torsion codes, E-codes, E-torsion graphs, graph representation, quasi-self dual codes

1. Introduction

Linear codes, well-studied objects in coding theory, have traditionally been explored over fields or rings with unity. However, recent researches [2–4, 14] have unveiled a fascinating avenue of investigation by extending the study of linear codes to non-unital rings. For instance, Alahmadi, et al [1], introduced the notion of Quasi Self-Dual codes (QSD codes), self-orthogonal linear codes of length n over a non-unital ring E such that the size of the code is 2^n. Moreover, there are some interesting researches in binary codes in the literature, for instance, [15] explored the Z_2-triple cycle codes and their duals, [11] cyclic codes from a sequence over finite fields, and [6] studied self-dual codes over R_k and binary self-dual codes. In continuation to the codes over E, Shi, Minjia, et al. [14] presented a special construction of QSD codes over E, based on combinatorial matrices

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related to two-class association schemes, Strongly Regular Graphs (SRG), and Doubly Regular Tournaments (DRT).

In this article, we delved into the analysis of graphs generated from linear codes over E, called linear E-codes and examine their properties and use these concepts to formulate a definition of graph.

Graph theory provides a powerful framework for visualizing and understanding complex systems, making it an ideal tool for investigating linear codes over non-unital rings. By associating codes with corresponding graphs, we can gain insights into the structure and behavior of these codes, enabling us to extract valuable information related to error correction, network coding, and other areas of interest. For standard notations and concepts in graph theory, the readers are advised to refer to [9].

In this study, we will first establish the foundations of linear codes over E, elucidating the necessary definitions, properties, and construction methods. Next, we will introduce the graph representation of such linear codes, by defining (k_1, k_2) E-torsion graph of an E-code, and will discuss the construction of such graphs and explore the relationship between the code's properties and the resulting graph structure. Moreover, we will study vertex-weighted graph to separate the isomorphic graph generated by two inequivalent E-codes.

The study of coding theory in relation to graph theory is not well-established topic. However, few researchers tried to focus on the subject such as graph theoretic methods in coding theory [13], where it discusses the application of graph theory in coding theory, and codes on graphs [8], where it developed a fundamental theory of realizations of linear and group codes on general graphs using elementary group theory, including basic group duality theory.

Through our comprehensive analysis of graphs produced from linear codes over the non-unital ring E, this article seeks to contribute to the expanding field of coding theory and its applications in diverse domains. By exploring the interplay between graph theory and linear codes over non-unital rings, we strive to unlock new perspectives, insights, and practical solutions that can address challenges in error correction, information transmission, and beyond.

2. Background

2.1. Binary codes

As defined in [14], denoted by $wt(x)$ the Hamming weight of $x \in \mathbb{F}_2^n$. The **dual** of a binary code C is denoted by $C^\perp$ and defined as

$$C^\perp = \{y \in \mathbb{F}_2^n | \forall x \in C, (x, y) = 0, \}$$

where

$$(x, y) = \sum_{i=1}^{n} x_i y_i,$$

denotes the standard inner product. A code C is **self-orthogonal** if it is included in its dual:

$$C \subseteq C^{\perp}.$$

Two binary codes are **equivalent** if there is a permutation of coordinates that maps one to the other.

2.2. Ring Theory

We describe the main properties of the ring E of order four. The ring E is defined by the relations on two generators a, b and we shall write

$$c = a + b$$

for the given ring.

The ring E is defined by

$$E = \langle a, b | 2a = 2b = 0, a^2 = a, b^2 = b, ab = a, ba = b\rangle.$$

It is a non-unital ring and non-commutative ring with characteristic two. For more details refer to [3, 7, 12]. The ring is local with maximal ideal $\{0, c\}$. Its multiplication table is given in Table 1.

×	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	b	b	0
c	0	c	c	0

Table 1: Multiplication table for the ring E

From Table 1, it is clear E is not commutative, and non-unital. It is local with the maximal ideal

$$J = \{0, c\},$$

and residue field

$$E/J = \mathbb{F} = \{0, 1\},$$

the finite filed of order 2.

If we denote

$$\alpha : E \to E/J = \mathbb{F}_2,$$

the map of reduction modulo J. It follows that

$$\alpha(0) = \alpha(c) = 0,$$

and

$$\alpha(a) = \alpha(b) = 1.$$

This function α is extended in the natural way in a map from E^n to $\mathbb{F}_2^n$. Readers who wanted further details on the properties of ring $\mathcal{R}$, we refer the readers to [1–3, 10].

2.3. Codes over E

A **linear** E-code of length n is a one-sided E-submodule of E^n. Let C be a code of length n over E. With the code, there are two binary codes of length n:

(i) the **residue code** defined by $res(C) = \{\alpha(y)|y \in C\}$,

(ii) the **torsion code** defined by $tor(C) = \{x \in \mathbb{F}_2^n|cx \in C\}$.

The **right dual** $C^{\perp_R}$ of C is the right module defined by

$$C^{\perp_R} = \{y \in E^n|\forall x \in C, (x, y) = 0\}.$$

The **left dual** $C^{\perp_R}$ of C is the left module defined by

$$C^{\perp_L} = \{y \in E^n|\forall x \in C, (y, x) = 0\}.$$

An E-code C is **self-orthogonal** if

$$\forall x, y \in C, (x, y) = 0.$$

It follows that C is **self-orthogonal** if and only if

$$C \subseteq C^{\perp_L}.$$

Similarly, C is **self-orthogonal** if and only if

$$C \subseteq C^{\perp_R}.$$

Hence, for a self-orthogonal code C, it satisfies that

$$C \subseteq C^{\perp_L} \cap C^{\perp_R}.$$

An $E-$code of length n is **Quasi Self-Dual** (QSD for short) [14] if it is self-orthogonal and of size 2^n. A quasi-self dual code is **Type IV** if all its codewords have even weight [5].

3. Some results in linear E-codes

3.1. Linear E-codes

Definition 1. *[3] Let C be a linear E-code. Then C is a **type-**(k_1, k_2) code if*

$$dim(res(C)) = k_1$$

and

$$dim(tor(C)) = k_1 + k_2.$$

Theorem 1. *[3] Let B be a self-orthogonal binary code of length n. The code C defined by the relation*

$$C = aB + cB^{\perp},$$

is a quasi self-dual code. Its residue code is B and its torsion code is $B^{\perp}$.

Corollary 1. *[3] Let B and B' be a binary code of length n such that B is self-orthogonal and $B \subseteq B'$. Then C is a linear E-code defined by the relation*

$$C = aB + cB'.$$

4. Results in (k_1, k_2) E-torsion graph of an E-code

Definition 2. *Let C be a linear E-code and B' be the torsion code of C. Then the simple graph G_{EC} such that the vertex set*

$$V(G_{EC}) = B'$$

and

$$\overline{xy} \in E(G_{EC}),$$

the edge set and $x \neq y$, if

$$ax + cy \in C$$

or

$$ay + cx \in C,$$

*is called the (k_1, k_2) E-**torsion graph** of C.*

To avoid the confusion to whether the binary code is viewed as a codeword in $tor(C)$ or vertex in G_{EC}, we denote the vertex $\widehat{x}$ which corresponds to the codeword x. This means that if

$$x \in tor(C),$$

then

$$\widehat{x} \in V(G_{EC}).$$

Example 1. *Let*

$$C = aB + cB'$$

where

$$B = \langle 1100 \rangle$$

and

$$B' = \langle 1100, 0011 \rangle.$$

This means that

$$V(G_{EC}) = \{\widehat{0000}, \widehat{1100}, \widehat{0011}, \widehat{1111}\}.$$

By computation, we get

$$E(G_{EC}) = \{(\widehat{0000}, \widehat{1100}), (\widehat{0000}, \widehat{0011}), (\widehat{0000}, \widehat{1111}), (\widehat{1100}, \widehat{0011}), (\widehat{1100}, \widehat{1111})\}.$$

Thus, the (k_1, k_2)-torsion graph of C, G_{EC}, is illustrated in Figure 1.

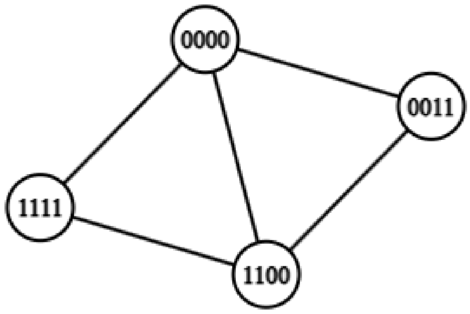

Figure 1: (k_1, k_2) E-torsion graph of C

Theorem 2. *If C is a type-(k_1, k_2) of an E-code, then*

$$|V(G_{EC})| = 2^{k_1+k_2}$$

and

$$|E(G_{EC})| = \sum_{i=1}^{2^{k_1}} 2^{k_1+k_2} - i.$$

Proof. The equation

$$|V(G_{EC})| = 2^{k_1+k_2}$$

follows from the fact that the torsion of a type-(k_1, k_2) E-code has dimension $k_1 + k_2$. On the other hand, from the definition of $E(G_{EC})$,

$$E(G_{EC}) = \{(\widehat{x}, \widehat{y}) : x \in res(C), y \in tor(C)\},$$

that is, each of the 2^{k_1} elements of the residue will be connected by an edge to the

$$2^{k_1+k_2} - 1$$

elements of the torsion. We can enumerate the edges by starting at an element in the residue with $2^{k_1+k_2} - 1$ edges containing that element, then if there is another element of the residue, we will enumerate the $2^{k_1+k_2} - 2$ edges containing the second element, since there is one edge common to the set of edges containing the first element and set of edges containing the second element, hence the second set of edges is 1 less than the previous set of edges. We continue the process by subtracting 1 from the number of the previous set of edges. Using this algorithm, the number of distinct pairs would be

$$\sum_{i=1}^{2^{k_1}} 2^{k_1+k_2} - i.$$

■

Corollary 2. *Let $\widehat{x} \in V(G_{EC})$. If $x \in res(C)$, then*

$$deg(\widehat{x}) = 2^{k_1+k_2} - 1.$$

If $x \notin res(C)$, then

$$deg(\widehat{x}) = 2^{k_1}.$$

Proof. The proof follows from Theorem 2. ∎

Corollary 3. *If C is a type-(k_1, k_2) E-code, then*

$$|E(G_{EC})| = 2^{2k_1+k_2} - 2^{2k_1-1} - 2^{k_1-1}.$$

Proof. The proof follows directly from Corollary 2. ∎

Lemma 1. $r(G_{EC}) = 1$.

Proof. If $x \in res(C)$, then the eccentricity of $\widehat{x}$ is 1 since $\widehat{x}$ is connected by an edge to every vertex in G_{EC}. If $x \notin res(C)$, then the eccentricity of $\widehat{x}$ is 2 since every vertex in G_{EC} is connected through a vertex in $res(C)$ to all other vertex not in $res(C)$. Therefore,

$$r(G_{EC}) = 1.$$

∎

Lemma 2. *Let $G_{EC} \neq P_2$, path of order 2. If there exists $x \notin res(C)$, then there exists $y \neq x$ such that $y \notin res(C)$.*

Proof. Let $x \notin res(C)$. Then

$$|res(C)| < |tor(C)|.$$

This means $k_1 < k_1 + k_2$, that is, $k_2 > 0$. Now,

$$|tor(C)| - |res(C)| = 2^{k_1+k_2} - 2^{k_1} = 2^{k_1}\left(2^{k_2} - 1\right).$$

Note that if $k_1 = 0$ and $k_2 = 1$, $G_{EC} \neq P_2$, which is a contradiction. Thus,

$$2^{k_1}\left(2^{k_2} - 1\right) \geq 2.$$

∎

Theorem 3. *Let C be an E-code and G_{EC} be the (k_1, k_2) E-torsion graph of C which is not P_2. Then vertex $\widehat{x} \in C(G_{EC})$ if and only if $x \in res(C)$.*

Proof. Let $\widehat{x} \in C(G_{EC})$. Suppose $x \notin res(C)$. Then, by Lemma 2 there exists $y \in tor(C)$ such that both

$$ax + cy$$

and

$$ay + cx$$

not in C. It follows that eccentricity of $\widehat{x}$ is greater than 1, a contradiction that $\widehat{x} \in C(G_{EC})$ by Lemma 1.
Conversely, suppose $x \in res(C)$. Then $\widehat{x}$ is connected by an edge to every vertex in G_{EC}. Thus, the eccentricity of vertex $\widehat{x}$ is 1, that is, $\widehat{x} \in C(G_{EC})$. ■

4.1. (k_1, k_2) E-torsion graph of QSD codes

Quasi self-dual codes are classified in [3] using their residue codes. But since every residue code corresponds to a unique torsion code, the study of the structure of G_{EC} of a QSD code will be concentrated in this section.

Example 2. *Let*

$$C = aB + cB^{\perp},$$

where

$$B = \langle 1100, 0011 \rangle .$$

Then

$$B^{\perp} = \langle 1100, 0011 \rangle$$

By Theorem 1*, C is a QSD code.*

$$V(G_{EC}) = \{\widehat{0000}, \widehat{1100}, \widehat{0011}, \widehat{1111}\}.$$

By Corollary 3*,*

$$|E(G_{EC})| = 16 - 8 - 2 = 6,$$

that is, G_{EC} is a complete graph.

Theorem 4. *Let G_{EC} be the (k_1, k_2) E-torsion graph of a QSD code*

$$C = aB + cB^{\perp}$$

where B is a binary code. Then B is self-dual if and only if G_{EC} is a complete graph.

Proof. Let B be self-dual. Then

$$res(C) = tor(C).$$

By Corollary 2, the degree of every vertex of G_{EC} is

$$2^{k_1+k_2} - 1,$$

that is, G_{EC} is a complete graph.
Conversely, suppose that G_{EC} is a complete graph. Let $x \in tor(C)$. Then

$$(\widehat{x}, \widehat{y}) \in E(G_{EC})$$

since G_{EC} is complete. It follows that

$$ax + cy \in C$$

for all

$$y \in tor(C).$$

Applying α, we have $x \in res(C)$, that is,

$$tor(C) \subseteq res(C).$$

■

Corollary 4. *If C is a QSD code of type-$(k_1, 0)$, then G_{EC} is a complete graph.*

Theorem 5. *If C is a QSD code of type-$(0, k_2)$, then G_{EC} is a star graph.*

Proof. If $k_1 = 0$, then $res(C)$ is the trivial code which contains only the zero vector. It follows that

$$tor(C) = \mathbb{F}_2^n.$$

Hence,

$$E(G_{EC}) = \{(\widehat{0_v}, \widehat{x}) : x \in \mathbb{F}_2^n\}.$$

■

Remark 1. *Let $k_1, k_2 \in \mathbb{Z}^+$ and C_1, C_2 be type-(k_1, k_2) linear E-codes. Then*

$$G_{EC_1} \cong G_{EC_2}.$$

Looking at Remark 1, (k_1, k_2) E-torsion graph alone cannot be used to classify QSD codes since two inequivalent codes under the same type-(k_1, k_2) code have the same (k_1, k_2) E-torsion graph. So to separate these two inequivalent QSD codes, we use the concept of vertex-weighted graph which is defined in the following.

Definition 3. *The **vertex-weighted** (k_1, k_2) **E-torsion graph** of a QSD code is the vertex-weighted graph where the weight of a vertex $x \in G_{EC}$ is the weight of the codeword $wt(x)$ of $x \in tor(C)$.*

Example 3. *Let*

$$C_1 = aB_1 + cB_1^{\perp}$$

and

$$C_2 = aB_2 + cB_2^{\perp}$$

where

$$B_1 = \langle 1100 \rangle$$

and

$$B_2 = \langle 1111 \rangle .$$

Note that C_1 and C_2 are two nonequivalents E-codes. Now,

$$V(G_{EC_1}) = \{\widehat{0000}, \widehat{1100}, \widehat{0010}, \widehat{1110}, \widehat{0001}, \widehat{1101}, \widehat{0011}, \widehat{1111}\}$$

and

$$V(G_{EC_2}) = \{\widehat{0000}, \widehat{1111}, \widehat{1100}, \widehat{0011}, \widehat{0110}, \widehat{1001}, \widehat{1010}, \widehat{0101}\}.$$

Figure 2 shows the graph representation of G_{EC_1}:

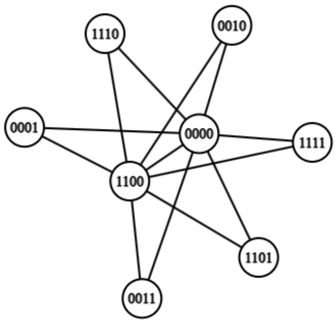

Figure 2: (k_1, k_2) E-torsion graph of G_{EC_1}

Furthermore, Figure 3 is the graph representation of graph G_{EC_2}.

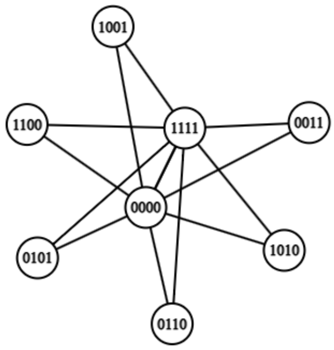

Figure 3: (k_1, k_2) E-torsion graph of G_{EC_2}

Note that the two graphs are isomorphic. However, if we look at the vertex-weighted graph of G_{EC_1} and G_{EC_2}, respectively, (see Figure 4 and 5) using the weights of every codeword, we see the difference between these two vertex-weighted $(1, 2)$ E-torsion graphs. Hence, two codes can have isomorphic graphs but different vertex-weighted (k_1, k_2) E-torsion graphs.

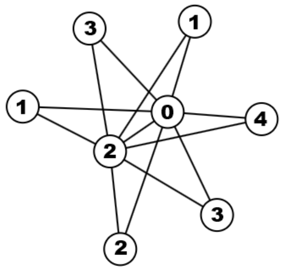

Figure 4: (k_1, k_2) E-torsion graph of G_{EC_1}

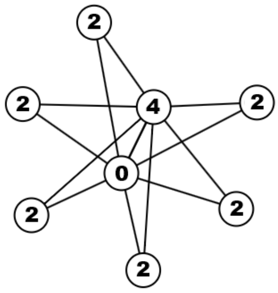

Figure 5: (k_1, k_2) E-torsion graph of G_{EC_2}

5. Vertex-weighted (k_1, k_2) E-torsion graph of QSD codes with $n \leq 4$

Quasi self-dual E-codes of short length were classified in [3]. In this section, we will illustrate those QSD codes using their vertex-weighted (k_1, k_2) E-torsion graphs up to $n = 4$.

5.1. (k_1, k_2) E-torsion graph of QSD codes for n=2.

For

$$C_1 = a \langle 00 \rangle + c \langle 10, 01 \rangle ,$$

we have a $(0, 2)$ E-torsion graph which is illustrated in Figure 6.

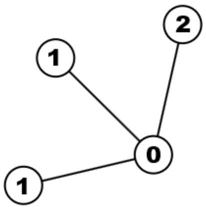

Figure 6: Vertex-weighted (k_1, k_2) E-torsion graph of C_1

For

$$C_2 = a\langle 11\rangle + c\langle 11\rangle,$$

we have a $(1,0)$ E-torsion graph which is illustrated in Figure 7.

Figure 7: Vertex-weighted (k_1, k_2) E-torsion graph of C_2

5.2. (k_1, k_2) E-torsion graph of QSD codes for n=3.

For

$$C_3 = a\langle 000\rangle + c\langle 100, 010, 001\rangle,$$

we have a $(0,3)$ E-torsion graph which is illustrated in Figure 8.

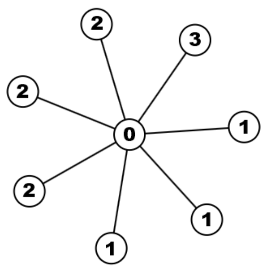

Figure 8: Vertex-weighted (k_1, k_2) E-torsion graph of C_3

5.3. (k_1, k_2) E-torsion graph of QSD codes for n=4.

For

$$C_5 = a\langle 0000\rangle + c\langle 1000, 0100, 0010, 0001\rangle,$$

we have a $(0,4)$ E-torsion graph which is illustrated in Figure 10.

For

$$C_4 = a\langle 101\rangle + c\langle 101, 010\rangle,$$

we have a $(1,1)$ E-torsion graph which is illustrated in Figure 9.

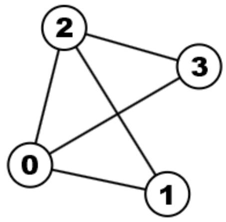

Figure 9: Vertex-weighted (k_1, k_2) E-torsion graph of C_4

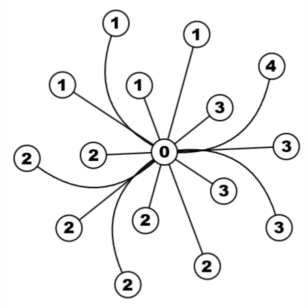

Figure 10: Vertex-weighted (k_1, k_2) E-torsion graph of C_5

For

$$C_6 = a\langle 1100\rangle + c\langle 1100, 0010, 0001\rangle,$$

we have a $(1,2)$ E-torsion graph which is illustrated in Figure 11.

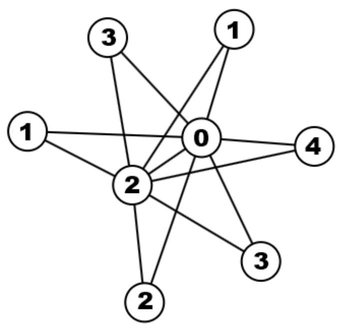

Figure 11: Vertex-weighted (k_1, k_2) E-torsion graph of C_6

For

$$C_7 = a\langle 1111\rangle + c\langle 1111, 1100, 0110\rangle,$$

we have a $(1, 2)$ E-torsion graph which is illustrated in Figure 12.

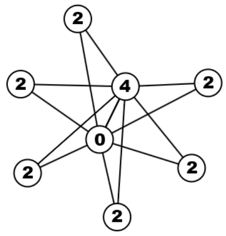

Figure 12: Vertex-weighted (k_1, k_2) E-torsion graph of C_7

For

$$C_8 = a\langle 1100, 0011\rangle + c\langle 1100, 0011\rangle,$$

we have a $(2, 0)$ E-torsion graph which is illustrated in Figure 13.

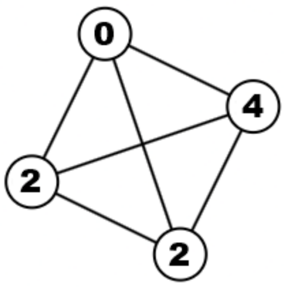

Figure 13: Vertex-weighted (k_1, k_2) E-torsion graph of C_8

6. Conclusion

In this paper, we studied the (k_1, k_2) E-torsion graph of a type-(k_1, k_2) E-codes. In particular, the size of the set of vertices and set of edges. We also characterized (k_1, k_2) E-torsion graph when $k_1 = 0$ and $k_2 = 0$ and introduced the notion of vertex-weighted (k_1, k_2) E-torsion graph to differentiate inequivalent QSD codes of the same type. Finally, we were able to represent QSD codes which were classified in [3] up to $n = 4$ using the vertex-weighted (k_1, k_2) E-torsion graph. By defining a (k_1, k_2) E-torsion graph G such that the $V(G) = 2^{k_1+k_2}$, there are 2^{k_1} vertices that have degree $2^{k_1+k_2} - 1$ with the rest vertices, if there exist, have degree 2^{k_1}. For future study, after graph operations of two (k_1, k_2) E-torsion graphs is a (k_1, k_2) E-torsion graph? Also, one can explore center of (k_1, k_2) E-torsion graphs and the dominating sets of (k_1, k_2) E-torsion graphs.

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